SHORT ANSWER TYPE

1. **Give an example of a statement P(n) which is true for all n ≥ 4 but P(1), P(2) and P(3) are not true. Justify your answer.**

Solution:

According to the question,
P(n) which is true for all n ≥ 4 but P(1), P(2) and P(3) are not true
Let P(n) be $2^n < n!$
So, the examples of the given statements are,
$P(0) \Rightarrow 2^0 < 0!$
i.e $1 < 1 \Rightarrow$ not true
$P(1) \Rightarrow 2^1 < 1!$
i.e $2 < 1 \Rightarrow$ not true
$P(2) \Rightarrow 2^2 < 2!$
i.e $4 < 2 \Rightarrow$ not true
$P(3) \Rightarrow 2^3 < 3!$
i.e $8 < 6 \Rightarrow$ not true
$P(4) \Rightarrow 2^4 < 4!$
i.e $16 < 24 \Rightarrow$ true
$P(5) \Rightarrow 2^5 < 5!$
i.e $32 < 60 \Rightarrow$ true, etc.

2. Give an example of a statement P(n) which is true for all n. Justify your answer.

Solution:

According to the question,
P(n) which is true for all n.
Let P(n) be

$$1 + 2 + 3 + \cdots \ldots \ldots + n = \frac{n(n+1)}{2}$$

$$P(0) \text{ is } 0 = \frac{0(0+1)}{2} = 0\,; \text{it's true}$$

$$P(1) \text{is } 1 = \frac{1(1+1)}{2} = 1\,; \text{it's true}$$

$$P(2) \text{is } 1 + 2 = \frac{2(2+1)}{2}\,; \text{it's true}$$

$$P(k) \text{ is } 1 + 2 + 3 + \cdots \ldots \ldots + k = \frac{k(k+1)}{2}$$

$$P(k) \text{is } 1 + 2 + 3 + \cdots \ldots \ldots + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$$

⇒ P(k) is true for all k.
Therefore, P(n) is true for all n.

Prove each of the statements in Exercises 3 to 16 by the Principle of Mathematical Induction:

3. $4^n – 1$ is divisible by 3, for each natural number n.

Solution:

According to the question,

$P(n) = 4^n – 1$ is divisible by 3.

So, substituting different values for n, we get,

$P(0) = 4^0 – 1 = 0$ which is divisible by 3.

$P(1) = 4^1 – 1 = 3$ which is divisible by 3.

$P(2) = 4^2 – 1 = 15$ which is divisible by 3.

$P(3) = 4^3 – 1 = 63$ which is divisible by 3.

Let $P(k) = 4^k – 1$ be divisible by 3,

So, we get,

$\Rightarrow 4^k – 1 = 3x$.

Now, we also get that,

$\Rightarrow \quad P(k+1) = 4^{k+1} – 1$

$= 4(3x + 1) – 1$

$= 12x + 3$ is divisible by 3.

$\Rightarrow$ P(k+1) is true when P(k) is true

Therefore, by Mathematical Induction,

$P(n) = 4^n – 1$ is divisible by 3 is true for each natural number n

4. $2^{3n} – 1$ is divisible by7, for all natural numbers n.

Solution:

According to the question,

$P(n) = 2^{3n} – 1$ is divisible by 7.

So, substituting different values for n, we get,

$P(0) = 2^0 – 1 = 0$ which is divisible by 7.

$P(1) = 2^3 – 1 = 7$ which is divisible by 7.

$P(2) = 2^6 – 1 = 63$ which is divisible by 7.

$P(3) = 2^9 - 1 = 512$ which is divisible by 7.

Let $P(k) = 2^{3k} - 1$ be divisible by 7

So, we get,

$\Rightarrow 2^{3k} – 1 = 7x$

Now, we also get that,

$\Rightarrow \quad P(k+1) = 2^{3(k+1)} – 1$

$= 2^3(7x + 1) – 1$

$= 56x + 7$

$= 7(8x + 1)$ is divisible by 7.

$\Rightarrow$ P(k+1) is true when P(k) is true.

Therefore, by Mathematical Induction,

$P(n) = 2^{3n} – 1$ is divisible by7, for all natural numbers n.

5. $n^3 – 7n + 3$ is divisible by 3, for all natural numbers n.

Solution:

According to the question,

$P(n) = n^3 – 7n + 3$ is divisible by 3.

So, substituting different values for n, we get,
P(0) = $0^3 - 7\times0 + 3 = 3$ which is divisible by 3.
P(1) = $1^3 - 7\times1 + 3 = -3$ which is divisible by 3.
P(2) = $2^3 - 7\times2 + 3 = -3$ which is divisible by 3.
P(3) = $3^3 - 7\times3 + 3 = 9$ which is divisible by 3.
Let P(k) = $k^3 - 7k + 3$ be divisible by 3
So, we get,
$\Rightarrow k^3 - 7k + 3 = 3x$.
Now, we also get that,

$$\begin{aligned}\Rightarrow \quad P(k+1) &= (k+1)^3 - 7(k+1) + 3\\ &= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3\\ &= 3x + 3(k^2 + k - 2) \text{ is divisible by 3.}\end{aligned}$$

$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,
P(n) = $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.

6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n.
Solution:

According to the question,
P(n) = $3^{2n} - 1$ is divisible by 8.
So, substituting different values for n, we get,
P(0) = $3^0 - 1 = 0$ which is divisible by 8.
P(1) = $3^2 - 1 = 8$ which is divisible by 8.
P(2) = $3^4 - 1 = 80$ which is divisible by 8.
P(3) = $3^6 - 1 = 728$ which is divisible by 8.
Let P(k) = $3^{2k} - 1$ be divisible by 8
So, we get,
$\Rightarrow 3^{2k} - 1 = 8x$.
Now, we also get that,

$$\begin{aligned}\Rightarrow \quad P(k+1) &= 3^{2(k+1)} - 1\\ &= 3^2(8x + 1) - 1\\ &= 72x + 8 \text{ is divisible by 8.}\end{aligned}$$

$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,
P(n) = $3^{2n} - 1$ is divisible by 8, for all natural numbers n.

7. For any natural number n, $7^n - 2^n$ is divisible by 5.
Solution:

According to the question,
P(n) = $7^n - 2^n$ is divisible by 5.
So, substituting different values for n, we get,
P(0) = $7^0 - 2^0 = 0$ Which is divisible by 5.
P(1) = $7^1 - 2^1 = 5$ Which is divisible by 5.
P(2) = $7^2 - 2^2 = 45$ Which is divisible by 5.
P(3) = $7^3 - 2^3 = 335$ Which is divisible by 5.
Let P(k) = $7^k - 2^k$ be divisible by 5

So, we get,
$\Rightarrow 7^k - 2^k = 5x$.
Now, we also get that,
$\Rightarrow \quad P(k+1) = 7^{k+1} - 2^{k+1}$
$= (5 + 2)7^k - 2(2^k)$
$= 5(7^k) + 2\,(7^k - 2^k)$
$= 5(7^k) + 2\,(5x)$ Which is divisible by 5.
$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,
$P(n) = 7^n - 2^n$ is divisible by 5 is true for each natural number n.

8. For any natural number n, $x^n - y^n$ is divisible by $x - y$, where x integers with $x \neq y$.
Solution:
According to the question,
$P(n) = x^n - y^n$ is divisible by $x - y$, x integers with $x \neq y$.
So, substituting different values for n, we get,
$P(0) = x^0 - y^0 = 0$ Which is divisible by $x - y$.
$P(1) = x - y$ Which is divisible by $x - y$.
$P(2) \quad = x^2 - y^2$
$= (x + y)(x - y)$ Which is divisible by $x - y$.
$P(3) \quad = x^3 - y^3$
$= (x - y)(x^2 + xy + y^2)$ Which is divisible by $x - y$.
Let $P(k) = x^k - y^k$ be divisible by $x - y$;
So, we get,
$\Rightarrow x^k - y^k = a(x - y)$.
Now, we also get that,
$\Rightarrow \quad P(k+1) = x^{k+1} - y^{k+1}$
$= x^k(x - y) + y(x^k - y^k)$
$= x^k(x - y) + y\,a(x - y)$ Which is divisible by $x - y$.
$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,
$P(n) = x^n - y^n$ is divisible by $x - y$, where x integers with $x \neq y$ which is true for any natural number n.

9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.
Solution:
According to the question,
$P(n) = n^3 - n$ is divisible by 6.
So, substituting different values for n, we get,
$P(0) = 0^3 - 0 = 0$ Which is divisible by 6.
$P(1) = 1^3 - 1 = 0$ Which is divisible by 6.
$P(2) = 2^3 - 2 = 6$ Which is divisible by 6.
Let $P(k) = k^3 - k$ be divisible by 6.
So, we get,

$\Rightarrow k^3 - k = 6x$.

Now, we also get that,

$\Rightarrow \quad P(k+1) = (k+1)^3 - (k+1)$

$= (k+1)(k^2+2k+1-1)$

$= k^3 + 3k^2 + 2k$

$= 6x+3k(k+1)$ [n(n+1) is always even and divisible by 2]

$= 6x + 3\times2y$ Which is divisible by 6.

$\Rightarrow$ P(k+1) is true when P(k) is true.

Therefore, by Mathematical Induction,

$P(n) = n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

10. $n(n^2 + 5)$ is divisible by 6, for each natural number n.

Solution:

According to the question,

$P(n) = n(n^2 + 5)$ is divisible by 6.

So, substituting different values for n, we get,

$P(0) = 0(0^2 + 5) = 0$ Which is divisible by 6.

$P(1) = 1(1^2 + 5) = 6$ Which is divisible by 6.

$P(2) = 2(2^2 + 5) = 18$ Which is divisible by 6.

$P(3) = 3(3^2 + 5) = 42$ Which is divisible by 6.

Let $P(k) = k(k^2 + 5)$ be divisible by 6.

So, we get,

$\Rightarrow k(k^2 + 5) = 6x.$

Now, we also get that,

$\Rightarrow \quad P(k+1) = (k+1)((k+1)^2 + 5) = (k+1)(k^2+2k+6)$

$= k^3 + 3k^2 + 8k + 6$

$= 6x+3k^2+3k+6$

$= 6x+3k(k+1)+6$[n(n+1) is always even and divisible by 2]

$= 6x + 3\times2y + 6$ Which is divisible by 6.

$\Rightarrow$ P(k+1) is true when P(k) is true.

Therefore, by Mathematical Induction,

$P(n) = n(n^2 + 5)$ is divisible by 6, for each natural number n.

11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.

Solution:

According to the question,

P(n) is $n^2 < 2^n$ for $n \geq 5$

Let $P(k) = k^2 < 2^k$ be true;

$\Rightarrow P(k+1) = (k+1)^2$

$= k^2 + 2k + 1$

$2^{k+1} = 2(2^k) > 2k^2$

Since, $n^2 > 2n + 1$ for $n \geq 3$

We get that,

$k^2 + 2k + 1 < 2k^2$

$\Rightarrow (k+1)^2 < 2^{(k+1)}$

$\Rightarrow$ P(k+1) is true when P(k) is true.

Therefore, by Mathematical Induction,

$P(n) = n^2 < 2^n$ is true for all natural numbers $n \geq 5$.

12. 2n < (n + 2)! for all natural number n.
Solution:

According to the question,
P(n) is $2n < (n + 2)!$
So, substituting different values for n, we get,
$P(0) \Rightarrow 0 < 2!$
$P(1) \Rightarrow 2 < 3!$
$P(2) \Rightarrow 4 < 4!$
$P(3) \Rightarrow 6 < 5!$
Let $P(k) = 2k < (k + 2)!$ is true;
Now, we get that,
$\Rightarrow P(k+1) = 2(k+1)\ ((k+1)+2))!$
We know that,
$[(k+1)+2)! = (k+3)! = (k+3)(k+2)(k+1)\ldots\ldots\ldots\ldots\ldots 3\times2\times1]$
But, we also know that,
$= 2(k+1) \times (k+3)(k+2)\ldots\ldots\ldots\ldots\ldots 3\times1 > 2(k+1)$
Therefore, $2(k+1) < ((k+1) + 2)!$
$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,
$P(n) = 2n < (n + 2)!$ Is true for all natural number n.

13. √n < 1/√1 + 1/√2 + … 1/√n, for all natural numbers n ≥ 2.
Solution:

According to the question,

$P(n)$ is $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots\ldots\ldots + \frac{1}{\sqrt{n}}$; $n \geq 2$ P(n) is $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots\ldots\ldots + \frac{1}{\sqrt{n}}$; $n \geq 2$

$P(2)$ is $\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \Rightarrow 1.414 < 1.707$ It's true

$P(3)$ is $\sqrt{3} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \Rightarrow 1.732 < 2.284$ It's true

Let $P(k)$ is $\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots\ldots\ldots + \frac{1}{\sqrt{k}}$; is true

Adding $\sqrt{k+1} - \sqrt{k}$ on both sides.

$$\Rightarrow \sqrt{k} + \sqrt{k+1} - \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots\ldots\ldots + \frac{1}{\sqrt{k}} + \sqrt{k+1} - \sqrt{k}$$

$$\left[\because \sqrt{k+1} - \sqrt{k} = \frac{\left(\sqrt{k+1} - \sqrt{k}\right)\left(\sqrt{k+1} + \sqrt{k}\right)}{\left(\sqrt{k+1} + \sqrt{k}\right)} = \frac{1}{\left(\sqrt{k+1} + \sqrt{k}\right)} \leq \frac{1}{\sqrt{k+1}}\right]$$

$$\Rightarrow \sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots\ldots\ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,

$\sqrt{n} < 1/\sqrt{1} + 1/\sqrt{2} + \dots 1/\sqrt{n}$, for all natural numbers $n \geq 2$

14. $2 + 4 + 6 + \dots + 2n = n^2 + n$ for all natural numbers n.

Solution:

According to the question,
P(n) is $2 + 4 + 6 + \dots + 2n = n^2 + n$.
So, substituting different values for n, we get,
$P(0) = 0 = 0^2 + 0$ Which is true.
$P(1) = 2 = 1^2 + 1$ Which is true.
$P(2) = 2 + 4 = 2^2 + 2$ Which is true.
$P(3) = 2 + 4 + 6 = 3^2 + 2$ Which is true.
Let $P(k) = 2 + 4 + 6 + \dots + 2k = k^2 + k$ be true;
So, we get,
$\Rightarrow$ P(k+1) is $2 + 4 + 6 + \dots + 2k + 2(k+1) = k^2 + k + 2k + 2$
$= (k^2 + 2k + 1) + (k+1)$
$= (k + 1)^2 + (k + 1)$
$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,
$2 + 4 + 6 + \dots + 2n = n^2 + n$ is true for all natural numbers n.

15. $1 + 2 + 2^2 + \dots 2^n = 2^{n+1} - 1$ for all natural numbers n.

Solution:

According to the question,
P(n) is $1 + 2 + 2^2 + \dots 2^n = 2^{n+1} - 1$.
So, substituting different values for n, we get,
$P(0) = 1 = 2^{0+1} - 1$ Which is true.
$P(1) = 1 + 2 = 3 = 2^{1+1} - 1$ Which is true.
$P(2) = 1 + 2 + 2^2 = 7 = 2^{2+1} - 1$ Which is true.
$P(3) = 1 + 2 + 2^2 + 2^3 = 15 = 2^{3+1} - 1$ Which is true.
Let $P(k) = 1 + 2 + 2^2 + \dots 2^k = 2^{k+1} - 1$ be true;
So, we get
$\Rightarrow$ P(k+1) is $1 + 2 + 2^2 + \dots 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$
$= 2 \times 2^{k+1} - 1$
$= 2^{(k+1)+1} - 1$

$\Rightarrow$ P(k+1) is true when P(k) is true.
Therefore, by Mathematical Induction,
$1 + 2 + 2^2 + \dots 2^n = 2^{n+1} - 1$ is true for all natural numbers n.