EXERCISE PAGE NO: 296

SHORT ANSWER TYPE:

1. Find the term independent of x, x ≠ 0, in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^{15}$.

Solution:

Given $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^{15}$

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{15}C_r\left(\frac{3x^2}{2}\right)^{15-r}\left(-\frac{1}{3x}\right)^r$$

$$T_{r+1} = {}^{15}C_r(-1)^r\, 3^{15-2r}2^{r-15}\, x^{30-3r}$$

For the term independent of x, we have

$30 - 3r = 0$

Which implies r = 10

By substituting the value of r in above obtained expression we get

$$T_{10+1} = {}^{15}C_{10}\, 3^{-5}\, 2^{-5}$$

$$= {}^{15}C_{10}\left(\frac{1}{6}\right)^5$$

2. If the term free from x in the expansion of $\sqrt{x}-\frac{k}{x^2}$ ^{10} is 405, find the value of k.

Solution:

Given $\sqrt{x}-\frac{k}{x^2}$ ^{10}

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r = {}^{10}C_r(x)^{\frac{1}{2}(10-r)}(-k)^r x^{-2r}$$

$$= {}^{10}C_r (x)^{5-\frac{r}{2}-2r} (-k)^r = {}^{10}C_r x^{\frac{10-5r}{2}} (-k)^r$$

For the term free from x we have

(10 – 5r)/2 = 0

Which implies r = 2

So, the term free from x is

$$T_{2+1} = {}^{10}C_2 (-k)^2$$

$$^{10}C_2 (-k)^2 = 405$$

$$\frac{10 \times 9 \times 8!}{2! \times 8!}(-k)^2 = 405$$

$$45k^2 = 405 \quad \Rightarrow \quad k^2 = 9 \quad \therefore k = \pm 3$$

3. Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$.

Solution:

Given $(1 - 3x + 7x^2)(1 - x)^{16}$

$$= (1 - 3x + 7x^2)({}^{16}C_0 - {}^{16}C_1 x^1 + {}^{16}C_2 x^2 + \ldots + {}^{16}C_{16} x^{16})$$

$$= (1 - 3x + 7x^2)(1 - 16x + 120x^2 + \ldots)$$

Coefficient of x = -19

4. Find the term independent of x in the expansion of $\left(3x - \frac{2}{x^2}\right)^{15}$

Solution:

Given $\left(3x - \frac{2}{x^2}\right)^{15}$

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{15}C_r (3x)^{15-r} \left(\frac{-2}{x^2}\right)^r = {}^{15}C_r 3^{15-r} x^{15-3r} (-2)^r$$

For the term independent of x, we have

15 – 3r = 0

Which implies r = 5

The term independent of x is

$$T_{5+1} = {}^{15}C_5\, 3^{15-5}\, (-2)^5$$

$$= \frac{15\times14\times13\times12\times11\times10!}{5\times4\times3\times2\times1\times10!} \cdot 3^{10} \cdot 2^5$$

$$= -3003 \times 3^{10} \times 2^5$$

5. Find the middle term (terms) in the expansion of

(i) $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$ (ii) $\left(3x - \frac{x^3}{6}\right)^{9}$

Solution:

a. Given $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$

Here index n = 10 which is even number.

So, there is one middle term which is $(10/2 + 1)^{th}$ term that is 6^{th} term

$$\therefore \quad T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$

$$= -{}^{10}C_5 \left(\frac{x}{a}\right)^5 \left(\frac{a}{x}\right)^5$$

$$= -\frac{10\times9\times8\times7\times6\times5!}{5!\times5\times4\times3\times2\times1} \left(\frac{x}{a}\right)^5 \left(\frac{x}{a}\right)^{-5} = -252$$

b. Given $\left(3x - \frac{x^3}{6}\right)^9$

Here index n = 9 which is odd

So, there is one middle term which is $(9/2 + 1)^{th}$ term that is 5^{th} term and 6^{th} term

$$\therefore \quad T_5 = T_{4+1} = {}^9C_4\, (3x)^{9-4} \left(-\frac{x^3}{6}\right)^4$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} 3^5 x^5 x^{12} 6^{-4}$$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \frac{3^5}{3^4 \times 2^4} x^{17} = \frac{189}{8} x^{17}$$

And 6th term,

$$T_6 = T_{5+1} = {}^9C_5 (3x)^{9-5} \left(-\frac{x^3}{6}\right)^5$$

$$= -\frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1} \cdot 3^4 \cdot x^4 \cdot x^{15} \cdot 6^{-5} = -\frac{-21}{16} x^{19}$$

6. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.

Solution:

Given $(x - x^2)^{10}$

$$T_{r+1} = {}^{10}C_r x^{10-r} (-x^2)^r = (-1)^r {}^{10}C_r x^{10-r} x^{2r} = (-1)^r {}^{10}C_r x^{10+r}$$

For the coefficient of x^{15}, we have

$$10 + r = 15 \quad \Rightarrow \quad r = 5$$

$$T_{5+1} = (-1)^5 {}^{10}C_5 x^{15}$$

$$\text{Coefficient of } x^{15} = -\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5 \times 4 \times 3 \times 2 \times 1 \times 5!} = -252$$

7. Find the coefficient of $1/x^{17}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Solution:

Given $\left(x^4 - \frac{1}{x^3}\right)^{15}$

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{15}C_r (x^4)^{15-r} \left(-\frac{1}{x^3}\right)^r = {}^{15}C_r x^{60-4r} (-1)^r x^{-3r} = {}^{15}C_r x^{60-7r} (-1)^r$$

For the coefficient x^{-17}, we have

$60 - 7r = -17$

Therefore, r = 11

Then above expression becomes

$T_{11+1} = {}^{15}C_{11}\, x^{60-77}(-1)^{11}$

Coefficient of $x^{-17} = \dfrac{-15 \times 14 \times 13 \times 12 \times 11!}{11! \times 4 \times 3 \times 2 \times 1}$

$= -15 \times 7 \times 13 = -1365$

8. Find the sixth term of the expansion $\left(y^{\frac{1}{2}} + x^{\frac{1}{3}}\right)^n$ if the binomial coefficient of the third term from the end is 45.

Solution:

Given $(y^{1/2} + x^{1/3})^n$.

Also given that binomial coefficient of third term from the end = 45

Therefore,

${}^nC_{n-2} = 45$

The above expression can be written as

${}^nC_2 = 45$

$\Rightarrow \dfrac{n(n-1)(n-2)!}{2!(n-2)!} = 45$

$\Rightarrow n(n-1) = 90$

$\Rightarrow n^2 - n - 90 = 0$

$\Rightarrow (n-10)(n+9) = 0$

Therefore n = 10

Now, sixth term $= {}^{10}C_5\, (y^{1/2})^{10-5}\, (x^{1/3})^5 = 252\, y^{5/2} \cdot x^{5/3}$

9. Find the value of r, if the coefficients of $(2r + 4)^{th}$ and $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal.

Solution:

Given $(1 + x)^{18}$

Now, $(2r + 4)^{th}$ term,

That is $T_{(2r+3)+1}$

$$T_{(2r+3)+1} = {}^{18}C_{2r+3}\,(x)^{2r+3}$$

And $(r-2)^{th}$ term, that is $T_{(r-3)+1}$

$$T_{(r-3)+1} = {}^{18}C_{r-3}x^{r-3}$$

Now according to the question,

$$^{18}C_{2r+3} = {}^{18}C_{r-3}$$

$$2r + 3 + r - 3 = 18$$

$$3r = 18 \quad \therefore \quad r = 6$$

10. If the coefficient of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A.P. Show that $2n^2 - 9n + 7 = 0$.

Solution:

Given $(1 + x)^{2n}$

Now, coefficient of 2^{nd}, 3^{rd} and 4^{th} terms are ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$, respectively.

Given that, ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in A.P.

Then,

$$2 \cdot {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$$

$$2\left[\frac{2n(2n-1)(2n-2)!}{2\times 1\times(2n-2)!}\right] = 2n + \frac{2n(2n-1)(2n-2)(2n-3)!}{3!(2n-3)!}$$

$$n(2n-1) = n + \frac{n(2n-1)(n-1)}{3}$$

$$3(2n-1) = 3 + (2n^2 - 3n + 1)$$

$$6n - 3 = 2n^2 - 3n + 4 \quad \Rightarrow \quad 2n^2 - 9n + 7 = 0$$

11. Find the coefficient of x4 in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Solution:

Given expression is $(1 + x + x^2 + x^3)^{11}$

$= [(1 + x) + x^2(1 + x)]^{11} = [(1 + x)(1 + x^2)]^{11} = (1 + x)^{11} \cdot (1 + x^2)^{11}$

$= ({}^{11}C_0 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 + \ldots)({}^{11}C_0 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + \ldots)$

Coefficient of $x^4 = {}^{11}C_0 \times {}^{11}C_4 + {}^{11}C_1 \times {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_0$

$= 330 + 605 + 55 = 990$

LONG ANSWER TYPE:

12. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ Is 1120, find p.

Solution:

Given expansion is $\left(\frac{p}{2}+2\right)^8$

Since index is $n = 8$, there is only one middle term, i.e., $\left(\frac{8}{2}+1\right)$th $= 5^{\text{th}}$ term

$$T_5 = T_{4+1} = {}^8C_4\left(\frac{p}{2}\right)^{8-4} \cdot 2^4$$

$$\Rightarrow \quad 1120 = {}^8C_4 p^4$$

By substituting the values, we get

$$\Rightarrow \quad 1120 = \frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1}p^4$$

$$\Rightarrow \quad 1120 = 7\times2\times5\times p^4$$

$$\Rightarrow \quad p^4 = \frac{1120}{70}$$

$$\Rightarrow \quad p^4 = 16$$

$$\Rightarrow \quad p^2 = 4 \quad \Rightarrow \quad p = \pm 2$$

13. Show that the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$ is

$$\frac{1\times3\times5\times\ldots\times(2n-1)}{n!}\times(-2)^n$$

Solution:

Given, expression is $\left(x-\frac{1}{x}\right)^{2n}$.

Since the index is $2n$, which is even. So, there is only one middle term, i.e.,

$\left(\frac{2n}{2}+1\right)$th term = $(n+1)$th term

$$T_{n+1} = {}^{2n}C_n(x)^{2n-n}\left(-\frac{1}{x}\right)^n = {}^{2n}C_n(-1)^n = (-1)^n\frac{(2n!)}{n!\cdot n!}$$

$$= (-1)^n\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\ldots(2n-1)\cdot(2n)}{n!\cdot n!} = (-1)^n\frac{[1\cdot 3\cdot 5\ldots(2n-1)]\cdot[2\cdot 4\cdot 6\ldots(2n)]}{(1\cdot 2\cdot 3\ldots n)\cdot n!}$$

$$= (-1)^n\frac{[1\cdot 3\cdot 5\ldots(2n-1)]\cdot 2^n[1\cdot 2\cdot 3\ldots n]}{(1\cdot 2\cdot 3\ldots n)\cdot n!} = (-2)^n\frac{[1\cdot 3\cdot 5\ldots(2n-1)]\cdot 2^n}{n!}$$

14. Find n in the binomial $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, if the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$.

Solution:

Given expression is $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$

Now, 7th term from beginning, $T_7 = T_{6+1} = {}^nC_6(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6$

And 7th term from end is same as 7th term from the beginning of $\left(\frac{1}{\sqrt[3]{3}}+\sqrt[3]{2}\right)^n$

$$T_7 = {}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}(\sqrt[3]{2})^6$$

Given that

$$\frac{{}^nC_6(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6}{{}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}(\sqrt[3]{2})^6}=\frac{1}{6}$$

$$\frac{(\sqrt[3]{2})^{n-12}}{\left(\frac{1}{\sqrt[3]{3}}\right)^{n-12}}=\frac{1}{6} \Rightarrow (\sqrt[3]{2}\sqrt[3]{3})^{n-12}=6^{-1} \Rightarrow 6^{\frac{n-12}{3}}=6^{-1}$$

$$\frac{n-12}{3}=-1 \Rightarrow n=9$$

15. In the expansion of $(x + a)^n$ if the sum of odd terms is denoted by O and the sum of even term by E. Then prove that
(i) $O^2 - E^2 = (x^2 - a^2)^n$
(ii) $4OE = (x + a)^{2n} - (x - a)^{2n}$

Solution:
(i) We know that
$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + \dots$$
Sum of odd terms,
$$O = {}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \dots$$
And also sum of even terms
$$E = {}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \dots$$

Since $(x + a)^n = O + E$
$(x - a)^n = O - E$
Therefore,
$(O + E)(O - E) = (x + a)^n (x - a)^n$

(ii) $40E = (O + E)^2 - (O - E)^2$
$= (x - a)^{2n} - (x - a)^{2n}$

16. If x^p occurs in the expansion of $x^2 + \frac{1}{x}$ ^{2n}
Prove that its coefficient is

$$\frac{\lfloor 2n}{\left\lfloor \frac{4n-p}{3} \right. \left\lfloor \frac{2n+p}{3} \right.}.$$

Solution:

Given expression is $\left(x^2+\frac{1}{x}\right)^{2n}$

Using the standard formula above expression can be written as

$$T_{r+1} = {}^{2n}C_r (x^2)^{2n-r}\left(\frac{1}{x}\right)^r = {}^{2n}C_r x^{4n-3r}$$

If x^p occurs in the expansion,

Let $4n - 3r = p$

$$3r = 4n - p \quad \Rightarrow \quad r = \frac{4n-p}{3}$$

$$\text{Coefficient of } x^p = {}^{2n}C_r = \frac{(2n)!}{r!\,(2n-r)!} = \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(2n-\frac{4n-p}{3}\right)!}$$

$$= \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$$

17. Find the term independent of x in the expansion of

$$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

Solution:

Given

$$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

Consider

$$\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

Using standard formula above expression can be written as

$$T_{r+1} = {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r = {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$$

Hence the general term in the expression of given expansion is

$$ {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r} + {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{19-3r}$$

$$+2\cdot {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{21-3r}$$

For independent term of x, substitute 18 – 3r = 0

19 – 3r = 0 and 21 – 3r = 0

We get r = 6 and r = 7

Hence second term is not independent of x

Therefore, term independent of x is

$$ {}^9C_6\left(\frac{3}{2}\right)^{9-6}\left(-\frac{1}{3}\right)^6 + 2\cdot {}^9C_7\left(\frac{3}{2}\right)^{9-7}\left(-\frac{1}{3}\right)^7$$

$$=\frac{9\times8\times7\times6!}{6!\times3\times2}\cdot\frac{1}{2^3\cdot3^3}-2\cdot\frac{9\times8\times7!}{7!\times2\times1}\cdot\frac{3^2}{2^2}\cdot\frac{1}{3^7}$$

$$=\frac{84}{8}\cdot\frac{1}{3^3}-\frac{36}{4}\cdot\frac{2}{3^5}=\frac{21-4}{54}=\frac{17}{54}$$

OBJECTIVE TYPE QUESTIONS:

Choose the correct answer from the given options in each of the Exercises 18 to 24 (M.C.Q.).

18. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is

(A) 50 (B) 202 (C) 51 (D) none of these

Solution:

(C) 51

Explanation:

Given $(x + a)^{100} + (x - a)^{100}$

$$= ({}^{100}C_0x^{100} + {}^{100}C_1x^{99}a + {}^{100}C_2x^{98}a^2 + \ldots)$$
$$+ ({}^{100}C_0x^{100} - {}^{100}C_1x^{99}a + {}^{100}C_2x^{98}a^2 + \ldots)$$
$$= 2({}^{100}C_0x^{100} + {}^{100}C_2x^{98}a^2 + \ldots + {}^{100}C_{100}a^{100})$$

So, there are 51 terms
Hence option c is the correct answer.

19. Given the integers r > 1, n > 2, and coefficients of $(3r)^{th}$ and $(r + 2)^{nd}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then
(A) n = 2r (B) n = 3r (C) n = 2r + 1 (D) none of these

Solution:
(A) n = 2r

Explanation:
Given $(1 + x)^{2n}$

$$T_{3r} = T_{(3r-1)+1} = {}^{2n}C_{3r-1}\,x^{3r-1}$$
$$T_{r+2} = T_{(r+1)+1} = {}^{2n}C_{r+1}x^{r+1}$$
$${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$$
$$3r - 1 + r + 1 = 2n$$
$$n = 2r$$

Hence option A is the correct answer.

20. The two successive terms in the expansion of $(1 + x)^{24}$ whose coefficients are in the ratio 1: 4 are
(A) 3^{rd} and 4^{th} (B) 4^{th} and 5^{th} (C) 5^{th} and 6^{th} (D) 6^{th} and 7^{th}

Solution:
(C) 5^{th} and 6^{th}

Explanation:
Let the two successive terms in the expansion of $(1 + x)^{24}$ be $(r + 1)^{th}$ and $(r + 2)^{th}$ term
Now,
$$T_{r+1} = {}^{24}C_rx^r \text{ and } T_{r+2} = {}^{24}C_{r+1}x^{r+1}$$

Given

$$\frac{{}^{24}C_r}{{}^{24}C_{r+1}} = \frac{1}{4}$$

$$\Rightarrow \frac{\dfrac{(24)!}{r!(24-r)!}}{\dfrac{(24)!}{(r+1)!(24-r-1)!}} = \frac{1}{4} \Rightarrow \frac{(r+1)r!(23-r)!}{r!(24-r)(23-r)!} = \frac{1}{4} \Rightarrow \frac{r+1}{24-r} = \frac{1}{4}$$

4r + 4 = 24 – r

Which implies r = 4

$T_{4+1} = T_5$ and $T_{4+2} = T_6$

Hence the correct option is c