Short Answer (S.A.)

Using the properties of determinants in Exercises 1 to 6, evaluate:

1. $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

Solution:

Given, $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

[Applying $C_1 \to C_1 - C_2$]

$$= \begin{vmatrix} x^2-2x+2 & x-1 \\ 0 & x+1 \end{vmatrix}$$

$= (x^2 - 2x + 2) . (x + 1) - (x - 1) . 0$

$= x^3 - 2x^2 + 2x + x^2 - 2x + 2$

$= x^3 - x^2 + 2$

2. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

Solution:

Given, $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}$$

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$$

$$= (a+x+y+z)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2(a+z+x+y)$$

3. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Solution:

Given, $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

[Taking x^2, y^2 and z^2 common from C_1, C_2 and C_3, respectively]

$$= x^2y^2z^2\begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_3$]

$$= x^2y^2z^2\begin{vmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{vmatrix} = x^2y^2z^2\,(x(yz+yz))$$

$$= x^2y^2z^2 \cdot (2xyz) = 2x^3y^3z^3$$

4. $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Solution:

Given, $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

[Taking $(x+y+z)$ common from column C_1]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_3$]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 3y & x-y \\ 0 & -3z & 2z+x \end{vmatrix}$$

[Expanding along first column]

$= (x + y + z) . 1[3y(3z + x) + (3z)(x - y)]$
$= (x + y + z)(3yz + 3yx + 3xz)$
$= 3(x + y + z)(xy + yz + zx)$

5. $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

Solution:

Given, $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

$= \begin{vmatrix} 3x+4 & 3x+4 & 3x+4 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$= (3x+4)\begin{vmatrix} 1 & 1 & 1 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

$= (3x+4)\begin{vmatrix} 0 & 0 & 1 \\ -4 & 4 & x \\ 0 & -4 & x+4 \end{vmatrix} = 16(3x+4)$

[Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$]

6. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Solution:

Given, $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

[Applying $R_1 \to R_1 + R_2 + R_3$]

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

[Taking $(a+b+c)$ common from the first row]

$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

[Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$]

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & -(a+b+c) & 2b \\ a+b+c & a+b+c & c-a-b \end{vmatrix}$$

Lastly, expanding along R1, we have

$= (a + b + c) [1 \times 0 + (a + b + c)^2]$

$= (a + b + c)^3$

Using the properties of determinants in Exercises 7 to 9, prove that:

7. $$\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$$

Solution:

From the given,

[Multiplying R_1, R_2, R_3 by x, y, z respectively]

$$= \frac{1}{xyz}\begin{vmatrix} xy^2z^2 & xyz & xy+xz \\ x^2yz^2 & xyz & yz+xy \\ x^2y^2z & xyz & xz+yz \end{vmatrix}$$

Next

[Taking (xyz) common from C_1 and C_2]

$$= \frac{1}{xyz}(xyz)^2\begin{vmatrix} yz & 1 & xy+xz \\ xz & 1 & yz+xy \\ xy & 1 & xz+yz \end{vmatrix}$$

Then,

[Applying $C_3 \to C_3 + C_1$]

$$= xyz\begin{vmatrix} yz & 1 & xy+yz+zx \\ xz & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix}$$

Lastly,

[Taking $(xy + yz + zx)$ common from C_3]

$$= xyz\,(xy+yz+zx)\begin{vmatrix} yz & 1 & 1 \\ xz & 1 & 1 \\ xy & 1 & 1 \end{vmatrix}$$

$= 0$ [$\because$ C_2 and C_3 are identical]

8. $$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

Solution:

Given,
$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$= \begin{vmatrix} 2(y+z) & z & y \\ 2(z+x) & z+x & x \\ 2(y+x) & x & x+y \end{vmatrix}$$

Now,
$$= 2\begin{vmatrix} y+z & z & y \\ z+x & z+x & x \\ x+y & x & x+y \end{vmatrix}$$

[Applying $C_1 \to C_1 - C_2$]

Next,
$$= 2\begin{vmatrix} y & z & y \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix}$$

[Applying $C_3 \to C_3 - C_1$]

Lastly,
$$= 2\begin{vmatrix} y & z & 0 \\ 0 & z+x & x \\ y & x & x \end{vmatrix}$$

[Applying $R_3 \to R_3 - R_1$]

$$= 2\begin{vmatrix} y & z & 0 \\ 0 & z+x & x \\ 0 & x-z & x \end{vmatrix}$$

$$= 2y[(z+x)x - x(x-z)] = 2y[2xz] = 4xyz$$

9. $$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

Solution:

Given, $$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

[Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$]

$$= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Now,

[Taking $(a-1)$ common from R_1 and R_2]

$$(a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Finally,

[Expanding along R_3]

$$= (a-1)^2 [1 \cdot (a+1) - 2] = (a-1)^3$$

10. If A + B + C = 0, then prove that $$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

Solution:

Given, $$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$$

On finding the determinant, we have

$$= 1(1-\cos^2 A) - \cos C(\cos C - \cos A \cdot \cos B) + \cos B(\cos C \cdot \cos A - \cos B)$$
$$= \sin^2 A - \cos^2 C + \cos A \cdot \cos B \cdot \cos C + \cos A \cdot \cos B \cdot \cos C - \cos^2 B$$
$$= \sin^2 A - \cos^2 B + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$
$$= -\cos(A+B) \cdot \cos(A-B) + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$
$$[\because \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B)]$$

$= -\cos(-C)\cdot\cos(A-B) + \cos C(2\cos A\cdot\cos B - \cos C)$
$= -\cos C(\cos A\cdot\cos B + \sin A\cdot\sin B - 2\cos A\cdot\cos B + \cos C)$
$= \cos C(\cos A\cdot\cos B - \sin A\cdot\sin B - \cos C)$
$= \cos C[\cos(A+B) - \cos C]$
$= \cos C(\cos C - \cos C)$ (As $\cos C = \cos(A+B)$)
$= 0$

11. If the co-ordinates of the vertices of an equilateral triangle with sides of length '*a*' are (x_1, y_1), (x_2, y_2), (x_3, y_3), then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \dfrac{3a^4}{4}$.

Solution:

We know that, the area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Also, we know the area of an equilateral triangle with side a is given by

$$\Delta = \frac{\sqrt{3}}{4}a^2$$

Hence,

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{4}a^2$$

On squaring both the sides, we get

$$\Rightarrow \Delta^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{16}a^4$$

or $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}$$

12. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$.

Solution:

Given,

$$\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$$

On expanding along C_3, we have

$\sin 3\theta \times (28-21) - \cos 2\theta \times (-7-7) - 2(3+4) = 0$

$7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$

$\sin 3\theta + 2 \cos 2\theta - 2 = 0$

$(3 \sin \theta - 4 \sin^3 \theta) + 2(1 - 2 \sin^2 \theta) - 2 = 0$

$4 \sin^3\theta - 4 \sin^2 \theta + 3 \sin \theta = 0$

$\sin \theta(4 \sin^2 \theta - 4 \sin \theta + 3) = 0$

$\sin \theta(4 \sin^2 \theta - 6 \sin \theta + 2 \sin \theta + 3) = 0$

$\sin \theta(2 \sin \theta + 1)(2 \sin \theta - 3) = 0$

$\sin \theta = 0$ or $\sin \theta = -1/2$ or $\sin \theta = 3/2$

$$\theta = n\pi \text{ or } \theta = m\pi + (-1)^n \left(-\frac{\pi}{6}\right); m, n \in Z$$

$$\sin \theta = \frac{-3}{2} \text{ is not possible}$$

13. If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix}$ **, then find values of *x*.**

Solution:

Given, $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

[Applying $R_1 \to R_1 + R_2 + R_3$], we have

$$\Rightarrow \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

Now,

[Taking $(12+x)$ common from R_1]

$$\Rightarrow \quad (12+x)\begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

Next,

[Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$]

$$\Rightarrow \quad (12+x)\begin{vmatrix} 0 & 0 & 1 \\ 0 & -2x & 4+x \\ 2x & 2x & 4-x \end{vmatrix} = 0$$

$$\Rightarrow \quad (12+x)\,(0-(-2x)(2x)] = 0$$

$$(12+x)\,(4x^2) = 0$$

Hence, $x = -12, 0$

14. If a_1, a_2, a_3, ..., a_r are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$$ **is independent of r.**

Solution:

We know that,

$$a_{r+1} = AR^{(r+1)-1} = AR^r;$$

where a_r = rth term of G.P.,

A = First term of G.P.

and R = Common ratio of G.P.

Now, $$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix} = \begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$$

[Taking AR^r, AR^{r+6} and AR^{r-10} common from R_1, R_2 and R_3, respectively]

$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10}\begin{vmatrix} 1 & AR^4 & AR^8 \\ 1 & AR^4 & AR^8 \\ 1 & AR^6 & AR^{10} \end{vmatrix}$$

$= 0$ [As R_1 and R_2 are identical]

Hence, the determinant is independent of r.

15. Show that the points (*a* + 5, *a* – 4), (*a* – 2, *a* + 3) and (*a*, *a*) do not lie on a straight line for any value of *a*.
Solution:

Given points are (*a* + 5, *a* – 4), (*a* – 2, *a* + 3) and (*a*, *a*).
Now, we have to prove that these points do not lie on a straight line.
So, if we prove that these points form a triangle then it can't line on a straight line.

$$\text{Area, } \Delta = \frac{1}{2}\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$$

[Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$]

$$= \frac{1}{2}\begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix} = \frac{1}{2}[(1\cdot(15-8)] = \frac{7}{2} \neq 0$$

Hence, the given points form a triangle and can't lie on a straight line.

16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta = \begin{bmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{bmatrix} = 0.$$

Solution:

Given,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

[Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$]

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A-\cos C & \cos B-\cos C & 1+\cos C \\ \cos^2 A+\cos A-\cos^2 C-\cos C & \cos^2 B+\cos B-\cos^2 C-\cos C & \cos^2 C+\cos C \end{vmatrix} = 0$$

Now,

[Taking $(\cos A - \cos C)$ common from C_1 and $(\cos B - \cos C)$ common from C_2]

$$\Rightarrow \quad (\cos A - \cos C)(\cos B - \cos C) \times$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A+\cos C+1 & \cos B+\cos C+1 & \cos^2 C+\cos C \end{vmatrix} = 0$$

[Applying $C_1 \to C_1 - C_2$]

$\Rightarrow$ $(\cos A - \cos C)(\cos B - \cos C) \times$

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1+\cos C \\ \cos A - \cos B & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

So,

$$(\cos A - \cos C)(\cos B - \cos C)(\cos B - \cos A) = 0$$

$$\cos A = \cos C \text{ or } \cos B = \cos C \text{ or } \cos B = \cos A$$

$$A = C \text{ or } B = C \text{ or } B = A$$

Hence, ΔABC is an isosceles triangle.

17. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1} = (A^2 - 3I)/2$.

Solution:

Given,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Co-factors are:

$$A_{11} = -1, A_{12} = 1, A_{13} = 1,$$
$$A_{21} = 1, A_{22} = -1, A_{23} = 1,$$
$$A_{31} = 1, A_{31} = 1, A_{32} = 1\ A_{33} = -1$$

Now, $\text{adj}\, A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$|A| = 0 - 1(-1) + 1.1 = 2$$

Thus, $A^{-1} = \dfrac{\text{adj}\, A}{|A|} = \dfrac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Now,

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Hence,

$$\frac{A^2-3I}{2}=\frac{1}{2}\left\{\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right\}$$

$$=\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}=A^{-1}$$

Hence proved.

Long Answer (L.A.)

18. If A = $\begin{bmatrix}1&2&0\\-2&-1&-2\\0&-1&1\end{bmatrix}$ **, find A^{-1}.**

Using A^{-1}, solve the system of linear equations
$x-2y=10$, $2x-y-z=8$, $-2y+z=7$.
Solution:

Given,

$$A=\begin{bmatrix}1&2&0\\-2&-1&-2\\0&-1&1\end{bmatrix}$$

Co-factors are:

$$A_{11}=-3,\ A_{12}=2,\ A_{13}=2,$$
$$A_{21}=-2,\ A_{22}=1,\ A_{23}=1,$$
$$A_{31}=-4,\ A_{32}=2,\ A_{33}=3$$

Now,

$$\text{adj}A=\begin{bmatrix}-3&2&2\\-2&1&1\\-4&2&3\end{bmatrix}^T=\begin{bmatrix}-3&-2&-4\\2&1&2\\2&1&3\end{bmatrix}$$

$$|A|=1(-3)-2(-2)+0=1$$

Hence,

$$A^{-1}=\frac{\text{adj}A}{|A|}=\begin{bmatrix}-3&-2&-4\\2&1&2\\2&1&3\end{bmatrix}$$

Now, the system of linear equations are

$$x-2y=10,$$
$$2x-y-z=8$$

and, $-2y + z = 7$

Or $AX = B$

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

where, $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B + \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

Thus, $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

$\therefore\ x = 0, y = -5$ and $z = -3$

19. Using matrix method, solve the system of equations $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$, $2x - y + z = 2$.

Solution:

Given system of equations are:

$3x + 2y - 2z = 3$

$x + 2y + 3z = 6$ and

$2x - y + z = 2$

Or,

AX = B

So, $$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

Hence, $X = A^{-1}B$

Now, for A^{-1} the co-factors are

$A_{11} = 5, A_{12} = 5, A_{13} = -5,$

$A_{21} = 0, A_{22} = 7, A_{23} = 7,$

$A_{31} = 10, A_{32} = -11$ and $A_{33} = 4$

So,
$$\text{adj } A = \begin{bmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

$$|A| = 3(5) + 2(5) + (-2)(-5) = 35$$

Thus,
$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 15+20 \\ 15+42-22 \\ -15+42+8 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, $x = 1$, $y = 1$ and $z = 1$

20. Given $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, **find BA and use this to solve the system of equations $y + 2z = 7$, $x - y = 3$, $2x + 3y + 4z = 17$.**

Solution:

Given,
$$A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

Now,
$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I$$

Thus,
$$B^{-1} = \frac{A}{6} = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

Given system of equations are:

$x - y = 3$, $2x + 3y + 4z = 17$ and $y + 2z = 7$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Therefore,
$x = 2, y = -1$ and $z = 4$

21. If $a + b + c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, **then prove that $a = b = c$.**

Solution:

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

[Applying $R_1 \to R_1 + R_2 + R_3$]

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Now,
[Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$]

$$\Delta = (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-a & a \\ c-b & a-b & b \end{vmatrix}$$

[Expanding along R_1]

$$= (a+b+c)[1(b-a)(a-b) - (c-a)(c-b)]$$
$$= (a+b+c)(ba - b^2 - a^2 + ab - c^2 + cb + ac - ab)$$
$$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= \frac{-1}{2}(a+b+c)[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$$
$$= -\frac{1}{2}(a+b+c)[(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ac)]$$
$$= \frac{-1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Given, $\Delta = 0$

$$\frac{-1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]=0$$

$$(a-b)^2+(b-c)^2+(c-a)^2=0 \qquad [\because a+b+c \neq 0, \text{ given}]$$

$$a-b=b-c=c-a=0$$

$$a=b=c$$

22. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ **is divisible by** $a+b+c$ **and find the quotient.**

Solution:

$$\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

Now, [Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$]

$$\Delta = \begin{vmatrix} bc-a^2-ca+b^2 & ca-b^2-ab+c^2 & ab-c^2 \\ ca-b^2-ab+c^2 & ab-c^2-bc+a^2 & bc-a^2 \\ ab-c^2-bc+a^2 & bc-a^2-ca+b^2 & ca-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab-c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc-a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca-b^2 \end{vmatrix}$$

Next,

[Taking $(a+b+c)$ common from C_1 and C_2 each]

$$\Delta = (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab-c^2 \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

Then,

[Applying $R_1 \to R_1 + R_2 + R_3$]

$$\Delta = (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

Lastly,

[Expanding along R_1]

$$\Delta = (a+b+c)^2[ab+bc+ca-(a^2+b^2+c^2)][(c-b)(b-a)-(a-c)^2]$$
$$= (a+b+c)^2(ab+bc+ca-a^2-b^2-c^2)\times (bc-ac-b^2+ab-a^2-c^2+2ac)$$
$$= (a+b+c)[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2]$$

Therefore, given determinant is divisible by $(a+b+c)$ and quotient is

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2$$

23. If $x+y+z=0$, prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Solution:

Taking, L.H.S. $= \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$

[Expanding]

$$= xa(a^2yz - x^2bc) - yb(y^2ac - b^2xz) + zc(c^2xy - z^2ab)$$
$$= xyza^3 - x^3abc - y^3abc + b^3xyz + c^3xyz - z^3abc$$
$$= xyz(a^3+b^3+c^3) - abc(x^3+y^3+z^3)$$
$$= xyz(a^3+b^3+c^3) - abc(3xyz)$$

$[\because x+y+z=0 \Rightarrow x^3+y^3+z^3-3xyz]$

$$= xyz(a^3+b^3+c^3-3abc)$$

$$= xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \text{R.H.S}$$

Hence proved.

Objective Type Questions (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, **then, value of x is**

(A) 3 **(B)** ± 3 **(C)** ± 6 **(D)** 6

Solution:

Option (C) ± 6

From the given,
On equating the determinants, we have
$2x^2 - 40 = 18 + 14$
$2x^2 = 72$
$x^2 = 36$
Thus, $x = \pm 6$

25. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$

(A) $a^3 + b^3 + c^3$ (B) 3 bc (C) $a^3 + b^3 + c^3 - 3abc$ (D) none of these
Solution:

Option (C) $a^3 + b^3 + c^3 - 3abc$
Given,

$$\Delta = \begin{vmatrix} a-b & b+c & a \\ b-c & c+a & b \\ c-a & a+b & c \end{vmatrix}$$

$$[\text{Applying } C_1 \to C_1 - C_3]$$

$$= \begin{vmatrix} -b & b+c & a \\ -c & c+a & b \\ -a & a+b & c \end{vmatrix}$$

$$[\text{Applying } C_2 \to C_2 + C_1]$$

$$= \begin{vmatrix} -b & c & a \\ -c & a & b \\ -a & b & c \end{vmatrix} = -\begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$= -[b(ac - b^2) - c(c^2 - ab) + a(bc - a^2)]$$
$$= a^3 + b^3 + c^3 - 3abc$$

26. The area of a triangle with vertices (–3, 0), (3, 0) and (0, *k*) is 9 sq. units. The value of *k* will be
(A) 9 (B) 3 (C) – 9 (D) 6
Solution:

Option (B) 3
We know that, the area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

Area of triangle with vertices (–3, 0), (3, 0) and (0, k) is

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}\right| = 9 \qquad \text{(given)}$$

$$[-3(-k) - 0 + 1(3k)] = \pm 18$$
$$6k = \pm 18$$

Thus, $k = \pm \dfrac{18}{6} = \pm 3$

27. The determinant $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$ **equals**

(A) $abc\,(b–c)\,(c – a)\,(a – b)$
(B) $(b–c)\,(c – a)\,(a – b)$
(C) $(a + b + c)\,(b – c)\,(c – a)\,(a – b)$
(D) None of these

Solution:

Option (D)

Given, $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

Now, [Taking $(b - a)$ common from C_1 and C_3 each]

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

[Applying $C_2 \to C_2 + C_3$]

$$= (b-a)^2 \begin{vmatrix} b & b & c \\ a & a & b \\ c & c & a \end{vmatrix}$$

$= 0$ [as C_1 and C_2 are identical]

28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$ **= 0 in the interval -π/4 ≤ x ≤ π/4 is**

(A) 0 **(B) 2** **(C) 1** **(D) 3**

Solution:

Option (C) 1

Given,

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 2\cos x + \sin x & \cos x & \cos x \\ 2\cos x + \sin x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

$$(2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Now,

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$(2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$(2\cos x + \sin x)[1 \cdot (\sin x - \cos x)^2] = 0$ (expanding along C_1)

$(2\cos x + \sin x)(\sin x - \cos x)^2 = 0$

$2\cos x = -\sin x$ or $\sin x = \cos x$

$\tan x = -2$, which is not possible as for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$,
we get $-1 \le \tan x \le 1$.

or, $\tan x = 1$

Thus, $x = \frac{\pi}{4}$

Therefore, only one real root exist.

29. If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

(A) 0 (B) -1 (C) 1 (D) None of these

Solution:

Option (A) 0

Given,

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

On expanding the determinant , we get

$$\Delta = -1 + 2\cos A\cos B\cos C + \cos^2 A + \cos^2 B + \cos^2 C$$

Now, $2\cos^2 A + 2\cos^2 B + 2\cos^2 C$

$$= 1 + \cos 2A + 1 + \cos 2B + 1 + \cos 2C$$
$$= 3 + (\cos 2A + \cos 2B + \cos 2C)$$
$$= 3 + (\cos 2A + \cos 2B) + \cos 2C$$
$$= 3 + 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1$$
$$= 2 + 2\cos(\pi - C)\cos(A-B) + 2\cos^2 C$$
$$= 2 - 2\cos C\cos(A-B) + 2\cos^2 C$$
$$= 2 - 2\cos C[\cos(A-B) - \cos C]$$
$$= 2 - 2\cos C[\cos(A-B) - \cos\{\pi - (A+B)\}]$$
$$= 2 - 2\cos C[\cos(A-B) + \cos(A+B)]$$
$$= 2 - 4\cos A\cos B\cos C$$
$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$$

Thus, $\Delta = 0$