

Short Answer (S.A.)

1. Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$

Solution:

We know that, $y = f(x)$ will be continuous at $x = a$ if,

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^+} f(x)$$

Given: $f(x) = x^3 + 2x^2 - 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} (1+h)^3 + 2(1+h)^2 - 1 = 1 + 2 - 1 = 2$$

$$\lim_{x \rightarrow 1} f(x) = (1)^3 + 2(1)^2 - 1 = 1 + 2 - 1 = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} (1+h)^3 + 2(1+h)^2 - 1 = 1 + 2 - 1 = 2$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1^+} f(x) = 2.$$

Thus, $f(x)$ is continuous at $x = 1$.

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

2. $f(x) = \begin{cases} 3x+5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$

Solution:

Checking the continuity of the given function, we have

$$\lim_{x \rightarrow 2^+} f(x) = 3x + 5 = \lim_{h \rightarrow 0} 3(2+h) + 5 = 11$$

$$\lim_{x \rightarrow 2} f(x) = 3x + 5 = 3(2) + 5 = 11$$

$$\lim_{x \rightarrow 2^-} f(x) = x^2 = \lim_{h \rightarrow 0} (2-h)^2 = \lim_{h \rightarrow 0} (2)^2 + h^2 - 4h = (2)^2 = 4$$

Now, since $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$

Thus, $f(x)$ is discontinuous at $x = 2$.

3. $f(x) = \begin{cases} \frac{1 - \cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$ at $x = 0$

Solution:

Checking the right hand and left hand limits of the given function, we have

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \frac{1 - \cos 2x}{x^2} \\&= \lim_{h \rightarrow 0} \frac{1 - \cos 2(0 - h)}{(0 - h)^2} = \lim_{h \rightarrow 0} \frac{1 - \cos(-2h)}{h^2} \\&= \lim_{h \rightarrow 0} \frac{1 - \cos 2h}{h^2} \\&= \lim_{h \rightarrow 0} \frac{2 \sin^2 h}{h^2} \quad \left[\because 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \right] \\&= \lim_{h \rightarrow 0} \frac{2 \sin h}{h} \cdot \frac{\sin h}{h} = 2.1.1 = 2 \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \frac{1 - \cos 2x}{x^2} \\&= \lim_{h \rightarrow 0} \frac{1 - \cos 2(0 + h)}{(0 + h)^2} = \lim_{h \rightarrow 0} \frac{1 - \cos 2h}{h^2} \\&= \lim_{h \rightarrow 0} \frac{2 \sin^2 h}{h^2} = \frac{2 \sin h}{h} \cdot \frac{\sin h}{h} = 2.1.1 = 2\end{aligned}$$

$$\lim_{x \rightarrow 0} f(x) = 5$$

$$\text{As } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0} f(x)$$

Therefore, the given function $f(x)$ is discontinuous at $x = 0$.

4.
$$f(x) = \begin{cases} \frac{2x^2 - 3x - 2}{x - 2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases} \quad \text{at } x = 2$$

Solution:

The given function at $x \neq 2$ can be rewritten as,

$$\begin{aligned}f(x) &= \frac{2x^2 - 3x - 2}{x - 2} \\&= \frac{2x^2 - 4x + x - 2}{x - 2} = \frac{2x(x - 2) + 1(x - 2)}{x - 2} \\&= \frac{(2x + 1)(x - 2)}{x - 2} = 2x + 1\end{aligned}$$

Now,

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= 2x + 1 \\&= \lim_{h \rightarrow 0} 2(2 - h) + 1 = 4 + 1 = 5\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 2^+} f(x) &= 2x + 1 \\&= \lim_{h \rightarrow 0} 2(2 + h) + 1 = 4 + 1 = 5\end{aligned}$$

$$\lim_{x \rightarrow 2} f(x) = 5$$

$$\text{As } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} f(x) = 5$$

Thus, $f(x)$ is continuous at $x = 2$.

$$f(x) = \begin{cases} \frac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases} \quad \text{at } x = 4$$

5.

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\begin{aligned} \lim_{x \rightarrow 4^-} f(x) &= \frac{|x-4|}{2(x-4)} \quad \left[\begin{array}{l} \text{for } x < 4, |x-4| = -(x-4) \\ \text{for } x > 4, |x-4| = (x-4) \end{array} \right] \\ &= \lim_{h \rightarrow 0} \frac{-(4-h-4)}{2[4-h-4]} = \lim_{h \rightarrow 0} \frac{h}{-2h} = -\frac{1}{2} \end{aligned}$$

$$\lim_{x \rightarrow 4^+} f(x) = \frac{|x-4|}{2(x-4)} = \lim_{h \rightarrow 0} \frac{[4+h-4]}{2[4+h-4]} = \frac{1}{2}$$

$$\lim_{x \rightarrow 4} f(x) = 0$$

$$\therefore \lim_{x \rightarrow 4^-} f(x) \neq \lim_{x \rightarrow 4^+} f(x) \neq \lim_{x \rightarrow 4} f(x)$$

Thus, $f(x)$ is discontinuous at $x = 4$.

$$f(x) = \begin{cases} |x| \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

6.

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= |x| \cos \frac{1}{x} \\ &= \lim_{h \rightarrow 0} |0-h| \cos \frac{1}{(0-h)} = \lim_{h \rightarrow 0} h \cos \frac{1}{h} \\ &= 0 \quad \left[\because \cos \frac{1}{x} \text{ oscillate between } -1 \text{ and } 1 \right] \end{aligned}$$

$$\lim_{x \rightarrow 0^+} f(x) = |x| \cos \frac{1}{x}$$

$$= \lim_{h \rightarrow 0} |0 + h| \cos \frac{1}{(0 + h)} = \lim_{h \rightarrow 0} h \cdot \cos \frac{1}{h} = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = 0$$

Thus, the given function $f(x)$ is continuous at $x = 0$.

7.
$$f(x) = \begin{cases} |x-a| \sin \frac{1}{x-a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases} \quad \text{at } x = a$$

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\begin{aligned} \lim_{x \rightarrow a^-} f(x) &= |x-a| \sin \frac{1}{x-a} \\ &= \lim_{h \rightarrow 0} |a-h-a| \cdot \sin \frac{1}{a-h-a} = \lim_{h \rightarrow 0} h \cdot \sin \frac{1}{-h} \\ &= \lim_{h \rightarrow 0} -h \cdot \sin \frac{1}{h} \quad [\because \sin(-\theta) = -\sin \theta] \\ &= 0 \times [\text{a number oscillating between } -1 \text{ and } 1] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) &= |x-a| \sin \frac{1}{x-a} \\ &= \lim_{h \rightarrow 0} |a+h-a| \cdot \sin \frac{1}{a+h-a} = \lim_{h \rightarrow 0} h \cdot \sin \frac{1}{h} \\ &= 0 \times [\text{a number oscillating between } -1 \text{ and } 1] \end{aligned}$$

$$\lim_{x \rightarrow a} f(x) = 0$$

Now, as

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} f(x) = 0$$

Thus, the given function $f(x)$ is continuous at $x = 0$.

8.
$$f(x) = \begin{cases} \frac{1}{e^x}, & \text{if } x \neq 0 \\ 1 + e^x, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \frac{e^{1/x}}{1 + e^{1/x}} \\&= \lim_{h \rightarrow 0} \frac{e^{\frac{1}{0-h}}}{1 + e^{\frac{1}{0-h}}} = \lim_{h \rightarrow 0} \frac{e^{-1/h}}{1 + e^{-1/h}} \\&= \lim_{h \rightarrow 0} \frac{1}{e^{1/h} (1 + e^{-1/h})} = \lim_{h \rightarrow 0} \frac{1}{e^{1/h} - 1} = \frac{1}{e^{1/0} - 1} \\&= \frac{1}{e^\infty - 1} = \frac{1}{0 - 1} = -1 \quad [\because e^\infty = 0]\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \frac{e^{1/x}}{1 + e^{1/x}} \\&= \lim_{h \rightarrow 0} \frac{e^{\frac{1}{0+h}}}{1 + e^{\frac{1}{0+h}}} = \lim_{h \rightarrow 0} \frac{e^{1/h}}{1 + e^{1/h}} \\&= \lim_{h \rightarrow 0} \frac{1}{e^{-1/h} (1 + e^{1/h})} = \lim_{h \rightarrow 0} \frac{1}{e^{-1/h} + 1} \\&= \frac{1}{e^{-\infty} + 1} = \frac{1}{0 + 1} = 1 \quad [e^{-\infty} = 0]\end{aligned}$$

$$\lim_{x \rightarrow 0} f(x) = 0$$

Now, as

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0} f(x)$$

Thus, $f(x)$ is discontinuous at $x = 0$.

$$f(x) = \begin{cases} \frac{x^2}{2}, & \text{if } 0 \leq x \leq 1 \\ 2x^2 - 3x + \frac{3}{2}, & \text{if } 1 < x \leq 2 \end{cases} \quad \text{at } x = 1$$

9. Solution:

Checking the right hand and left hand limits for the given function, we have

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \frac{x^2}{2} = \lim_{h \rightarrow 0} \frac{(1-h)^2}{2} = \frac{1}{2} \\ \lim_{x \rightarrow 1^+} f(x) &= \frac{x^2}{2} = \frac{(1)^2}{2} = \frac{1}{2}\end{aligned}$$

$$\lim_{x \rightarrow 1^-} f(x) = 2x^2 - 3x + \frac{3}{2} = 2(1)^2 - 3(1) + \frac{3}{2} = 2 - 3 + \frac{3}{2} = \frac{1}{2}$$

Now, as

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = \frac{1}{2}$$

Thus, the given function $f(x)$ is continuous at $x = 1$.

10. $f(x) = |x| + |x - 1|$ at $x = 1$

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= |x| + |x - 1| = \lim_{h \rightarrow 0} |1 - h| + |1 - h - 1| \\ &= |1 - 0| + |1 - 0 - 1| = 1 + 0 = 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= |x| + |x - 1| \\ &= \lim_{h \rightarrow 0} |1 + h| + |1 + h - 1| = 1 + 0 = 1 \end{aligned}$$

$$\lim_{x \rightarrow 1} f(x) = |x| + |x - 1| = |1| + |1 - 1| = 1 + 0 = 1$$

Now, as

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x)$$

Thus, $f(x)$ is continuous at $x = 1$.

Find the value of k in each of the Exercises 11 to 14 so that the function f is continuous at the indicated point:

$$f(x) = \begin{cases} 3x - 8, & \text{if } x \leq 5 \\ 2k, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5$$

11.

Solution:

Finding the left hand and right hand limits for the given function, we have

$$\begin{aligned} \lim_{x \rightarrow 5^-} f(x) &= 3x - 8 \\ &= \lim_{h \rightarrow 0} 3(5 - h) - 8 = 15 - 8 = 7 \end{aligned}$$

$$\lim_{x \rightarrow 5^+} f(x) = 2k$$

As the function is continuous at $x = 5$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x)$$

So,

$$7 = 2k$$

$$k = 7/2 = 3.5$$

Therefore, the value of k is 3.5

$$f(x) = \begin{cases} \frac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \text{ at } x = 2$$

12.

Solution:

The given function $f(x)$ can be rewritten as,

$$f(x) = \frac{2^{x+2} - 16}{4^x - 16} = \frac{2^2 \cdot 2^x - 16}{(2^x)^2 - (4)^2} = \frac{4(2^x - 4)}{(2^x - 4)(2^x + 4)}$$

$$f(x) = \frac{4}{2^x + 4}$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{h \rightarrow 0} \frac{4}{2^{2+h} + 4} = \frac{4}{2^2 + 4} = \frac{4}{4 + 4} = \frac{4}{8} = \frac{1}{2}$$

$$\lim_{x \rightarrow 2} f(x) = k$$

As the function is continuous at $x = 2$.

$$\therefore \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} f(x)$$

So, $k = \frac{1}{2}$

Therefore, the value of k is $\frac{1}{2}$

$$f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \leq x \leq 1 \end{cases} \text{ at } x = 0$$

13.

Solution:

Finding the left hand and right hand limits for the given function, we have

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} \\ &= \lim_{x \rightarrow 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} \times \frac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}} \\ &= \lim_{x \rightarrow 0^-} \frac{(1+kx) - (1-kx)}{x[\sqrt{1+kx} + \sqrt{1-kx}]} \\ &= \lim_{x \rightarrow 0^-} \frac{1+kx - 1+kx}{x[\sqrt{1+kx} + \sqrt{1-kx}]} \\ &= \lim_{x \rightarrow 0^-} \frac{2kx}{x[\sqrt{1+kx} + \sqrt{1-kx}]} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0^+} \frac{2k}{\sqrt{1+kx} + \sqrt{1-kx}} \\
 &= \lim_{h \rightarrow 0} \frac{2k}{\sqrt{1+k(0-h)} + \sqrt{1-k(0-h)}} \\
 &= \frac{2k}{\sqrt{1} + \sqrt{1}} = \frac{2k}{2} = k
 \end{aligned}$$

$$\lim_{x \rightarrow 0^+} f(x) = \frac{2x+1}{x-1} = \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1$$

As the function is continuous at $x = 0$,

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = k = -1$$

Therefore, the value of k is -1

$$f(x) = \begin{cases} \frac{1 - \cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \frac{1}{2}, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

14.

Solution:

Finding the left hand and right hand limits for the given function, we have

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} f(x) &= \frac{1 - \cos kx}{x \sin x} \\
 &= \lim_{h \rightarrow 0} \frac{1 - \cos k(0-h)}{(0-h) \sin(0-h)} = \lim_{h \rightarrow 0} \frac{1 - \cos(-kh)}{-h \sin(-h)} \\
 &= \lim_{h \rightarrow 0} \frac{1 - \cos kh}{h \sin h} \quad \left[\begin{array}{l} \because \sin(-\theta) = -\sin \theta \\ \cos(-\theta) = \cos \theta \end{array} \right] \\
 &= \lim_{h \rightarrow 0} \frac{2 \sin^2 \frac{kh}{2}}{h \sin h} \\
 &= \lim_{\substack{h \rightarrow 0 \\ kh \rightarrow 0}} \frac{2 \sin \frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} \times \frac{\sin \frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} \times \frac{1}{h \cdot \frac{\sin h}{h} \cdot h} \\
 &= 2 \cdot 1 \cdot \frac{kh}{2} \cdot 1 \cdot \frac{kh}{2} \cdot \frac{1}{h^2} \cdot 1 \quad \left[\begin{array}{l} \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \text{ and} \\ \lim_{kh \rightarrow 0} \frac{\sin kh}{kh} = 1 \end{array} \right] \\
 &= \frac{k^2}{2}
 \end{aligned}$$

$$\lim_{x \rightarrow 0} f(x) = \frac{1}{2}$$

But, as the function is continuous we have

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

$$\text{So, } \frac{k^2}{2} = \frac{1}{2}$$

$$k^2 = 1 \Rightarrow k = \pm 1$$

Therefore, the value of k is ± 1

15. Prove that the function f defined by

$$f(x) = \begin{cases} \frac{x}{|x| + 2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of k .

Solution:

Finding the left hand and right hand limit for the given function, we have

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \frac{x}{|x| + 2x^2} = \lim_{h \rightarrow 0} \frac{0 - h}{|0 - h| + 2(0 - h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h + 2h^2} = \lim_{h \rightarrow 0} \frac{-h}{h(1 + 2h)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{1 + 2h} = \frac{-1}{1 + 2(0)} = -1 \\ \lim_{x \rightarrow 0^+} f(x) &= \frac{x}{|x| + 2x^2} = \lim_{h \rightarrow 0} \frac{0 + h}{|0 + h| + 2(0 + h)^2} \\ &= \lim_{h \rightarrow 0} \frac{h}{h + 2h^2} = \lim_{h \rightarrow 0} \frac{h}{h(1 + 2h)} = \frac{1}{1 + 0} = 1 \\ \lim_{x \rightarrow 0^-} f(x) &\neq \lim_{x \rightarrow 0^+} f(x) \end{aligned}$$

Now, as the left hand limit and the right hand limit are not equal and the value of both the limits are a constant.

Hence, regardless the choice of k , the given function remains discontinuous at $x = 0$.

16. Find the values of a and b such that the function f defined by

$$f(x) = \begin{cases} \frac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a+b, & \text{if } x = 4 \\ \frac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$$

is a continuous function at $x = 4$.

Solution:

Finding the left hand and right hand limit for the given function, we have

$$\begin{aligned}\lim_{x \rightarrow 4^-} f(x) &= \frac{x-4}{|x-4|} + a \\ &= \lim_{h \rightarrow 0} \frac{4-h-4}{|4-h-4|} + a = \lim_{h \rightarrow 0} \frac{-h}{h} + a = -1 + a \\ \lim_{x \rightarrow 4^-} f(x) &= a + b \\ \lim_{x \rightarrow 4^+} f(x) &= \frac{x-4}{|x-4|} + b \\ &= \lim_{h \rightarrow 0} \frac{4+h-4}{|4+h-4|} + b = \lim_{h \rightarrow 0} \frac{h}{h} + b = 1 + b\end{aligned}$$

As the function is continuous at $x = 4$.

$$\therefore \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4^+} f(x)$$

$$\text{So, } -1 + a = a + b = 1 + b$$

$$-1 + a = a + b \text{ and } 1 + b = a + b$$

$$\text{We get, } b = -1 \text{ and } 1 + -1 = a + -1 \Rightarrow a = 1$$

Therefore, the value of $a = 1$ and $b = -1$

17. Given the function $f(x) = 1/(x+2)$. Find the points of discontinuity of the composite function $y = f(f(x))$.

Solution:

Given,

$$\begin{aligned}f(x) &= \frac{1}{x+2} \\ f[f(x)] &= \frac{1}{f(x)+2} = \frac{1}{\frac{1}{x+2}+2} = \frac{1}{\frac{1+2x+4}{x+2}} = \frac{x+2}{2x+5} \\ \therefore f[f(x)] &= \frac{x+2}{2x+5}\end{aligned}$$

Now, the function will not be defined and continuous where

$$2x + 5 = 0 \Rightarrow x = -5/2$$

Therefore, $x = -5/2$ is the point of discontinuity.

18. Find all points of discontinuity of the function $f(t) = \frac{1}{t^2+t-2}$, where $t = \frac{1}{x-1}$.

Solution:

$$\begin{aligned}
 \text{Given, } f(t) &= \frac{1}{t^2 + t - 2} \\
 \Rightarrow f(t) &= \frac{1}{\frac{1}{(x-1)^2} + \frac{1}{(x-1)} - 2} \quad \left[\text{putting } t = \frac{1}{x-1} \right] \\
 &= \frac{1}{\frac{1 + x - 1 - 2(x-1)^2}{(x-1)^2}} = \frac{(x-1)^2}{x - 2x^2 - 2 + 4x} \\
 &= \frac{(x-1)^2}{-2x^2 + 5x - 2} = \frac{(x-1)^2}{-(2x^2 - 5x + 2)} \\
 &= \frac{(x-1)^2}{-[2x^2 - 4x - x + 2]} = \frac{(x-1)^2}{-[2x(x-2) - 1(x-2)]} \\
 &= \frac{(x-1)^2}{-(x-2)(2x-1)} = \frac{(x-1)^2}{(2-x)(2x-1)}
 \end{aligned}$$

Now,

if $f(t)$ is discontinuous, then $2 - x = 0 \Rightarrow x = 2$

And, $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

Therefore, the required points of discontinuity for the given function are 2 and $\frac{1}{2}$.

19. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$. Examine the differentiability of f , where f is defined by

Solution:

Given,

$f(x) = |\sin x + \cos x|$ at $x = \pi$

Now, put $g(x) = \sin x + \cos x$ and $h(x) = |x|$

Hence, $h[g(x)] = h(\sin x + \cos x) = |\sin x + \cos x|$

Now,

$g(x) = \sin x + \cos x$ is a continuous function since $\sin x$ and $\cos x$ are two continuous functions at $x = \pi$.

We know that, every modulus function is a continuous function everywhere.

Therefore, $f(x) = |\sin x + \cos x|$ is continuous function at $x = \pi$.

20.
$$f(x) = \begin{cases} x[x], & \text{if } 0 \leq x < 2 \\ (x-1)x, & \text{if } 2 \leq x < 3 \end{cases} \quad \text{at } x = 2.$$

Solution:

We know that, a function f is differentiable at a point 'a' in its domain if

$Lf'(c) = Rf'(c)$

where $Lf'(c) = \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h}$ and

$$Rf'(c) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Here, $f(x) = \begin{cases} x[x], & \text{if } 0 \leq x < 2 \\ (x-1)x, & \text{if } 2 \leq x < 2 \end{cases}$ at $x = 2$.

$$\begin{aligned} Lf'(c) &= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \rightarrow 0} \frac{(2-h)[2-h] - (2-1)2}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h) \cdot 1 - 2}{-h} \quad [\because [2-h] = 1] \\ &= \lim_{h \rightarrow 0} \frac{2-h-2}{-h} = 1 \end{aligned}$$

$$\begin{aligned} Rf'(c) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h-1)(2+h) - (2-1) \cdot 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)(2+h) - 2}{h} = \lim_{h \rightarrow 0} \frac{2+h+2h+h^2-2}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h+h^2}{h} = \lim_{h \rightarrow 0} \frac{h(3+h)}{h} = 3 \end{aligned}$$

$$Lf'(2) \neq Rf'(2)$$

Therefore, $f(x)$ is not differentiable at $x = 2$.

$$21. \quad f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

at $x = 0$.

Solution:

Given,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases} \quad \text{at } x = 0$$

For differentiability we know that:

$$Lf'(c) = Rf'(c)$$

$$\therefore Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{(0-h)^2 \sin \frac{1}{(0-h)} - 0}{-h} = \frac{h^2 \cdot \sin \left(-\frac{1}{h} \right)}{-h} \\
 &= h \cdot \sin \left(\frac{1}{h} \right) = 0 \times \left[-1 \leq \sin \left(\frac{1}{h} \right) \leq 1 \right] \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 Rf'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{(0+h)^2 \sin \left(\frac{1}{0+h} \right) - 0}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 \sin \left(\frac{1}{h} \right)}{h} = \lim_{h \rightarrow 0} h \cdot \sin \left(\frac{1}{h} \right) \\
 &= 0 \times \left[-1 \leq \sin \left(\frac{1}{h} \right) \leq 1 \right] = 0
 \end{aligned}$$

Hence, $Lf'(0) = Rf'(0) = 0$

Therefore, $f(x)$ is differentiable at $x = 0$

22.
$$f(x) = \begin{cases} 1+x & , \text{ if } x \leq 2 \\ 5-x & , \text{ if } x > 2 \end{cases}$$

at $x = 2$.

Solution:

We know that, $f(x)$ is differentiable at $x = 2$ if

$$Lf'(2) = Rf'(2)$$

Now,

$$\begin{aligned}
 Lf'(2) &= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} \\
 &= \lim_{h \rightarrow 0} \frac{(1+2-h) - (1+2)}{-h} = \lim_{h \rightarrow 0} \frac{3-h-3}{-h} = \frac{-h}{-h} = 1
 \end{aligned}$$

$$\begin{aligned}
 Rf'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[5-(2+h)] - (1+2)}{h} = \lim_{h \rightarrow 0} \frac{3-h-3}{h} \\
 &= \frac{-h}{h} = -1
 \end{aligned}$$

$$\text{So, } Lf'(2) \neq Rf'(2)$$

Thus, $f(x)$ is not differentiable at $x = 2$.

23. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

Solution:

Given, $f(x) = |x - 5|$

$$\Rightarrow f(x) = \begin{cases} -(x - 5) & \text{if } x - 5 < 0 \text{ or } x < 5 \\ x - 5 & \text{if } x - 5 > 0 \text{ or } x > 5 \end{cases}$$

For continuity at $x = 5$

$$\begin{aligned} \text{L.H.L. } \lim_{h \rightarrow 5^-} f(x) &= -(x - 5) \\ &= \lim_{h \rightarrow 0} -(5 - h - 5) = \lim_{h \rightarrow 0} h = 0 \end{aligned}$$

$$\begin{aligned} \text{R.H.L. } \lim_{x \rightarrow 5^+} f(x) &= x - 5 \\ &= \lim_{h \rightarrow 0} (5 + h - 5) = \lim_{h \rightarrow 0} h = 0 \end{aligned}$$

$$\text{L.H.L.} = \text{R.H.L.}$$

So, $f(x)$ is continuous at $x = 5$.

Now, for differentiability

$$\begin{aligned} \text{L}f'(5) &= \lim_{h \rightarrow 0} \frac{f(5 - h) - f(5)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{-(5 - h - 5) - (5 - 5)}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1 \end{aligned}$$

$$\begin{aligned} \text{R}f'(5) &= \lim_{h \rightarrow 0} \frac{f(5 + h) - f(5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(5 + h - 5) - (5 - 5)}{h} = \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1 \end{aligned}$$

Thus, $\text{L}f'(5) \neq \text{R}f'(5)$

Therefore, $f(x)$ is not differentiable at $x = 5$.

**24. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the equation $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}, f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.
Solution:**

Given,

$f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the equation $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}, f(x) \neq 0$

Let us take any point $x = 0$ at which the function $f(x)$ is differentiable.

$$\begin{aligned} \text{So, } f'(0) &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} \\ 2 &= \lim_{h \rightarrow 0} \frac{f(0) \cdot f(h) - f(0)}{h} \quad [\because f(0) = f(h)] \quad \dots(i) \end{aligned}$$

$$\Rightarrow 2 = \lim_{h \rightarrow 0} \frac{f(0)[f(h) - 1]}{h}$$

$$\begin{aligned} \text{Now, } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h} \quad [\because f(x + y) = f(x) \cdot f(y)] \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{f(x)[f(h)-1]}{h} = 2f(x) \quad \text{from eqn. (i)}$$

Therefore, $f'(x) = 2f(x)$.

Differentiate each of the following w.r.t. x (Exercises 25 to 43) :

25. $2^{\cos^2 x}$

Solution:

Let $y = 2^{\cos^2 x}$

Taking log on both sides, we get

$$\log y = \log 2^{\cos^2 x} \Rightarrow \log y = \cos^2 x \cdot \log 2$$

Now,

Differentiating both sides w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log 2 \cdot \frac{d}{dx} \cos^2 x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log 2 \left[2 \cos x \cdot \frac{d}{dx} \cos x \right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log 2 [2 \cos x (-\sin x)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log 2 (-\sin 2x)$$

$$\frac{dy}{dx} = -y \cdot \log 2 \sin 2x$$

Thus, $\frac{dy}{dx} = -2^{\cos^2 x} (\log 2 \sin 2x)$

26. $\frac{8^x}{x^8}$

Solution:

Let $y = \frac{8^x}{x^8}$

Taking log on both sides, we get $\log y = \log \frac{8^x}{x^8}$

$$\Rightarrow \log y = \log 8^x - \log x^8 \Rightarrow \log y = x \log 8 - 8 \log x$$

Differentiating both sides w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log 8 - \frac{8}{x} \Rightarrow \frac{dy}{dx} = y \left[\log 8 - \frac{8}{x} \right]$$

Thus, $\frac{dy}{dx} = \frac{8^x}{x^8} \left[\log 8 - \frac{8}{x} \right]$

27. $\log \left(x + \sqrt{x^2 + a} \right)$

Solution:

Let $y = \log \left(x + \sqrt{x^2 + a} \right)$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \log \left(x + \sqrt{x^2 + a} \right) \\ &= \frac{1}{x + \sqrt{x^2 + a}} \cdot \frac{d}{dx} \left(x + \sqrt{x^2 + a} \right) \\ &= \frac{1}{x + \sqrt{x^2 + a}} \cdot \left[1 + \frac{1}{2\sqrt{x^2 + a}} \times \frac{d}{dx} (x^2 + a) \right] \\ &= \frac{1}{x + \sqrt{x^2 + a}} \cdot \left[1 + \frac{1}{2\sqrt{x^2 + a}} \cdot 2x \right] \\ &= \frac{1}{x + \sqrt{x^2 + a}} \cdot \left[1 + \frac{x}{\sqrt{x^2 + a}} \right] \\ &= \frac{1}{x + \sqrt{x^2 + a}} \cdot \left(\frac{\sqrt{x^2 + a} + x}{\sqrt{x^2 + a}} \right) = \frac{1}{\sqrt{x^2 + a}} \end{aligned}$$

Thus, $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a}}$

28. $\log \left[\log (\log x^5) \right]$

Solution:

Let, $y = \log [\log (\log x^5)]$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \log [\log (\log x^5)] \\ &= \frac{1}{\log (\log x^5)} \times \frac{d}{dx} \log (\log x^5) \\ &= \frac{1}{\log (\log x^5)} \times \frac{1}{\log (x^5)} \times \frac{d}{dx} \log x^5 \\ &= \frac{1}{\log (\log x^5)} \cdot \frac{1}{\log (x^5)} \cdot \frac{1}{x^5} \cdot \frac{d}{dx} x^5 \end{aligned}$$

$$= \frac{1}{\log(\log x^5)} \cdot \frac{1}{\log(x^5)} \cdot \frac{1}{x^5} \cdot 5x^4$$

$$= \frac{5}{x \log(x^5) \cdot \log(\log x^5)}$$

Thus, $\frac{dy}{dx} = \frac{5}{x \log(x^5) \cdot \log(\log x^5)}$

29. $\sin \sqrt{x} + \cos^2 \sqrt{x}$

Solution:

Let $y = \sin \sqrt{x} + \cos^2 \sqrt{x}$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\sin \sqrt{x}) + \frac{d}{dx}(\cos^2 \sqrt{x}) \\ &= \cos \sqrt{x} \cdot \frac{d}{dx}(\sqrt{x}) + 2 \cos \sqrt{x} \cdot \frac{d}{dx}(\cos \sqrt{x}) \\ &= \cos \sqrt{x} \cdot \frac{1}{2\sqrt{x}} + 2 \cos \sqrt{x} (-\sin \sqrt{x}) \cdot \frac{d}{dx} \sqrt{x} \\ &= \frac{1}{2\sqrt{x}} \cdot \cos \sqrt{x} - 2 \cos \sqrt{x} \cdot \sin \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\cos \sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}} \end{aligned}$$

Thus, $\frac{dy}{dx} = \frac{\cos \sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$

30. $\sin^n(ax^2 + bx + c)$

Solution:

Let $y = \sin^n(ax^2 + bx + c)$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \sin^n(ax^2 + bx + c) \\ &= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \frac{d}{dx} \sin(ax^2 + bx + c) \\ &= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot \frac{d}{dx} (ax^2 + bx + c) \\ &= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot (2ax + b) \end{aligned}$$

Thus, $\frac{dy}{dx} = n(2ax + b) \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c)$

31. $\cos(\tan \sqrt{x+1})$

Solution:

Let $y = \cos(\tan \sqrt{x+1})$

Differentiating both sides w.r.t. x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \cos(\tan \sqrt{x+1}) \\ &= -\sin(\tan \sqrt{x+1}) \cdot \frac{d}{dx}(\tan \sqrt{x+1}) \\ &= -\sin(\tan \sqrt{x+1}) \cdot \sec^2 \sqrt{x+1} \cdot \frac{d}{dx} \sqrt{x+1} \\ &= -\sin(\tan \sqrt{x+1}) \cdot \sec^2 \sqrt{x+1} \cdot \frac{1}{2\sqrt{x+1}} \cdot 1\end{aligned}$$

Thus, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x+1}} \sin(\tan \sqrt{x+1}) \cdot \sec^2 \sqrt{x+1}$

32. $\sin x^2 + \sin^2 x + \sin^2(x^2)$

Solution:

Let $y = \sin x^2 + \sin^2 x + \sin^2(x^2)$

Differentiating both sides w.r.t. x ,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \sin(x^2) + \frac{d}{dx} \sin^2 x + \frac{d}{dx} \sin^2(x^2) \\ &= \cos x^2 \cdot \frac{d}{dx}(x^2) + 2 \sin x \cdot \frac{d}{dx}(\sin x) + 2 \sin(x^2) \cdot \frac{d}{dx} \sin(x^2) \\ &= \cos x^2 \cdot 2x + 2 \sin x \cdot \cos x + 2 \sin x^2 \cdot \cos x^2 \cdot \frac{d}{dx}(x^2) \\ &= 2x \cdot \cos x^2 + \sin 2x + 2 \sin x^2 \cdot \cos x^2 \cdot 2x\end{aligned}$$

Thus, $\frac{dy}{dx} = 2x \cdot \cos x^2 + \sin 2x + 2x \sin 2x^2$

33. $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Solution:

Let $y = \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) = \frac{1}{\sqrt{1 - \left(\frac{1}{\sqrt{x+1}}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{1}{\sqrt{1 - \frac{1}{x+1}}} \cdot \frac{d}{dx}(x+1)^{-1/2}$$

$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}} \cdot \frac{-1}{2}(x+1)^{-3/2} \cdot \frac{d}{dx}(x+1)$$

$$= \frac{\sqrt{x+1}}{\sqrt{x}} \cdot \frac{-1}{2}(x+1)^{-3/2} \cdot 1$$

$$= \frac{-1}{2} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} \cdot \frac{1}{(x+1)^{3/2}} = -\frac{1}{2\sqrt{x}(x+1)}$$

Thus, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}(x+1)}$

34. $(\sin x)^{\cos x}$

Solution:

Let $y = (\sin x)^{\cos x}$

Taking log on both sides,

$$\log y = \log (\sin x)^{\cos x}$$

$$\Rightarrow \log y = \cos x \cdot \log (\sin x)$$

$$[\because \log x^y = y \log x]$$

Differentiating both sides w.r.t. x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} \cos x \cdot \log (\sin x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \frac{d}{dx} \log (\sin x) + \log (\sin x) \cdot \frac{d}{dx} \cos x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) + \log (\sin x) \cdot (-\sin x)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \cot x \cdot \cos x - \sin x \cdot \log (\sin x)$$

$$\frac{dy}{dx} = y [\cot x \cdot \cos x - \sin x \cdot \log (\sin x)]$$

$$\text{Thus, } \frac{dy}{dx} = (\sin x)^{\cos x} \left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log(\sin x) \right]$$

35. $\sin^m x \cdot \cos^n x$

Solution:

$$\text{Let } y = \sin^m x \cdot \cos^n x$$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\sin^m x \cdot \cos^n x) \\ &= \sin^m x \cdot \frac{d}{dx}(\cos^n x) + \cos^n x \cdot \frac{d}{dx} \sin^m x \\ &= \sin^m x \cdot n \cdot \cos^{n-1} x \frac{d}{dx}(\cos x) + \cos^n x \cdot m \cdot \sin^{m-1} x \cdot \frac{d}{dx}(\sin x) \\ &= n \cdot \sin^m x \cdot \cos^{n-1} x \cdot (-\sin x) + m \cdot \cos^n x \cdot \sin^{m-1} x \cdot \cos x \\ &= -n \cdot \sin^{m+1} x \cdot \cos^{n-1} x + m \cdot \cos^{n+1} x \cdot \sin^{m-1} x \\ &= \sin^m x \cdot \cos^n x \left[-n \frac{\sin x}{\cos x} + m \cdot \frac{\cos x}{\sin x} \right] \end{aligned}$$

$$\text{Thus, } \frac{dy}{dx} = \sin^m x \cdot \cos^n x [-n \tan x + m \cdot \cot x]$$

36. $(x+1)^2 + (x+2)^3 + (x+3)^4$

Solution:

$$\text{Let } y = (x+1)^2(x+2)^3(x+3)^4$$

Taking log on both sides,

$$\log y = \log [(x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4]$$

$$\log y = \log (x+1)^2 + \log (x+2)^3 + \log (x+3)^4$$

$$[\because \log xy = \log x + \log y]$$

$$\Rightarrow \log y = 2 \log (x+1) + 3 \log (x+2) + 4 \log (x+3)$$

$$[\because \log x^y = y \log x]$$

Differentiating both sides w.r.t. x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{d}{dx} \log(x+1) + 3 \cdot \frac{d}{dx} \log(x+2) + 4 \cdot \frac{d}{dx} \log(x+3)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x+1} + 3 \cdot \frac{1}{x+2} + 4 \cdot \frac{1}{x+3}$$

$$\frac{dy}{dx} = y \left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3} \right]$$

Now,

$$\Rightarrow \frac{dy}{dx} = (x+1)^2(x+2)^3(x+3)^4 \left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3} \right]$$

$$= (x+1)^2(x+2)^3(x+3)^4 \left[\frac{2(x+2)(x+3) + 3(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)} \right]$$

$$= (x+1)(x+2)^2(x+3)^3(2x^2 + 10x + 12 + 3x^2 + 12x + 9 + 4x^2 + 12x + 8)$$

$$= (x+1)(x+2)^2(x+3)^3(9x^2 + 34x + 29)$$

Thus, $\frac{dy}{dx} = (x+1)(x+2)^2(x+3)^3(9x^2 + 34x + 29)$

37. $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{-\pi}{4} < x < \frac{\pi}{4}$

Solution:

Let $y = \cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$

$$= \cos^{-1}\left[\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right]$$

$$= \cos^{-1}\left[\sin \frac{\pi}{4} \sin x + \cos \frac{\pi}{4} \cdot \cos x\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{4} - x\right)\right]$$

$$y = \frac{\pi}{4} - x \quad \left[\because -\frac{\pi}{4} < x < \frac{\pi}{4} \right]$$

Differentiating both sides w.r.t. x

Thus, $\frac{dy}{dx} = -1$

38. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

Solution:

Let $y = \tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right]$

$$= \tan^{-1}\left[\sqrt{\frac{2 \sin^2 x/2}{2 \cos^2 x/2}}\right] \quad \left[\begin{array}{l} \because 1 - \cos x = 2 \sin^2 x/2 \\ 1 + \cos x = 2 \cos^2 x/2 \end{array} \right]$$

$$= \tan^{-1} \left[\frac{\sin x/2}{\cos x/2} \right] = \tan^{-1} \left[\tan \frac{x}{2} \right]$$

Thus, $y = \frac{x}{2}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2} \frac{d}{dx}(x) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Hence, $\frac{dy}{dx} = \frac{1}{2}$

$$\tan^{-1}(\sec x + \tan x), -\frac{\pi}{2} < x < \frac{\pi}{2}$$

39.

Solution:

Let $y = \tan^{-1}(\sec x + \tan x)$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(\sec x + \tan x)] \\ &= \frac{1}{1 + (\sec x + \tan x)^2} \cdot \frac{d}{dx} (\sec x + \tan x) \\ &= \frac{1}{1 + \sec^2 x + \tan^2 x + 2 \sec x \tan x} \cdot (\sec x \tan x + \sec^2 x) \\ &= \frac{1}{(1 + \tan^2 x) + \sec^2 x + 2 \sec x \tan x} \cdot \sec x (\tan x + \sec x) \\ &= \frac{1}{\sec^2 x + \sec^2 x + 2 \sec x \tan x} \cdot \sec x (\tan x + \sec x) \\ &= \frac{1}{2 \sec^2 x + 2 \sec x \tan x} \cdot \sec x (\tan x + \sec x) \\ &= \frac{1}{2 \sec x (\sec x + \tan x)} \cdot \sec x (\tan x + \sec x) = \frac{1}{2} \end{aligned}$$

Thus, $\frac{dy}{dx} = \frac{1}{2}$

$$\tan^{-1} \left(\frac{a \cos x - b \sin x}{b \cos x + a \sin x} \right), -\frac{\pi}{2} < x < \frac{\pi}{2} \text{ and } \frac{a}{b} \tan x > -1$$

40.

Solution:

$$\begin{aligned}\text{Let } y &= \tan^{-1} \left(\frac{a \cos x - b \sin x}{b \cos x + a \sin x} \right) \\ y &= \tan^{-1} \left[\frac{\frac{a \cos x}{b \cos x} - \frac{b \sin x}{b \cos x}}{\frac{b \cos x}{b \cos x} + \frac{a \sin x}{b \cos x}} \right] \\ y &= \tan^{-1} \left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b} \tan x} \right] \\ y &= \tan^{-1} \frac{a}{b} - \tan^{-1}(\tan x) \\ &\quad \left[\because \tan^{-1} \left(\frac{x-y}{1+xy} \right) = \tan^{-1} x - \tan^{-1} y \right]\end{aligned}$$

$$\Rightarrow y = \tan^{-1} \frac{a}{b} - x$$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\tan^{-1} \frac{a}{b} \right) - \frac{d}{dx}(x) = 0 - 1 = -1$$

Thus, $\frac{dy}{dx} = -1$.

41. $\sec^{-1} \left(\frac{1}{4x^3 - 3x} \right), 0 < x < \frac{1}{\sqrt{2}}$

Solution:

$$\text{Let } y = \sec^{-1} \left(\frac{1}{4x^3 - 3x} \right)$$

$$\text{Put } x = \cos \theta \quad \therefore \theta = \cos^{-1} x$$

$$\Rightarrow y = \sec^{-1} \left(\frac{1}{4 \cos^3 \theta - 3 \cos \theta} \right)$$

$$y = \sec^{-1} \left(\frac{1}{\cos 3\theta} \right) \quad [\because \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta]$$

$$y = \sec^{-1}(\sec 3\theta) \Rightarrow y = 3\theta$$

$$y = 3 \cos^{-1} x$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = 3 \cdot \frac{d}{dx} \cos^{-1} x = 3 \left(\frac{-1}{\sqrt{1-x^2}} \right) = \frac{-3}{\sqrt{1-x^2}}$$

Thus, $\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$.

42. $\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), \frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$

Solution:

Let $y = \tan^{-1} \left[\frac{3a^2x - x^3}{a^3 - 3ax^2} \right]$

Put $x = a \tan \theta \quad \therefore \theta = \tan^{-1} \frac{x}{a}$

$$y = \tan^{-1} \left[\frac{3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a \cdot a^2 \tan^2 \theta} \right]$$

$$y = \tan^{-1} \left[\frac{3a^3 \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a^3 \tan^2 \theta} \right]$$

$$y = \tan^{-1} \left[\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right]$$

$$y = \tan^{-1} [\tan 3\theta] \quad \left[\because \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right]$$

$$\Rightarrow y = 3\theta \Rightarrow y = 3 \tan^{-1} \frac{x}{a}$$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= 3 \cdot \frac{d}{dx} \left(\tan^{-1} \frac{x}{a} \right) \\ &= 3 \cdot \frac{1}{1 + \frac{x^2}{a^2}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right) = 3 \cdot \frac{a^2}{a^2 + x^2} \cdot \frac{1}{a} = \frac{3a}{a^2 + x^2} \end{aligned}$$

Thus, $\frac{dy}{dx} = \frac{3a}{a^2 + x^2}$.

43. $\tan^{-1} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right), -1 < x < 1, x \neq 0$

Solution:

$$\text{Let } y = \tan^{-1} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right)$$

$$\text{Putting } x^2 = \cos 2\theta \quad \therefore \theta = \frac{1}{2} \cos^{-1} x^2$$

$$y = \tan^{-1} \left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}} \right)$$

$$y = \tan^{-1} \left(\frac{\sqrt{2 \cos^2 \theta} + \sqrt{2 \sin^2 \theta}}{\sqrt{2 \cos^2 \theta} - \sqrt{2 \sin^2 \theta}} \right)$$

$$y = \tan^{-1} \left(\frac{\sqrt{2} \cos \theta + \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta - \sqrt{2} \sin \theta} \right)$$

$$y = \tan^{-1} \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right)$$

$$y = \tan^{-1} \left[\frac{\frac{\cos \theta}{\cos \theta} + \frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\cos \theta} - \frac{\sin \theta}{\cos \theta}} \right]$$

$$y = \tan^{-1} \left[\frac{1 + \tan \theta}{1 - \tan \theta} \right]$$

$$y = \tan^{-1} \left[\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \cdot \tan \theta} \right]$$

$$y = \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \theta \right) \right]$$

$$\Rightarrow y = \frac{\pi}{4} + \theta \Rightarrow y = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2$$

Differentiating both sides w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{4} \right) + \frac{1}{2} \frac{d}{dx} (\cos^{-1} x^2) \\ &= 0 + \frac{1}{2} \times \frac{-1}{\sqrt{1-x^4}} \cdot \frac{d}{dx} (x^2) = \frac{-1.2x}{2\sqrt{1-x^4}} = -\frac{x}{\sqrt{1-x^4}} \end{aligned}$$

$$\text{Thus, } \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^4}}$$

Find dy/dx of each of the functions expressed in parametric form in Exercises from 44 to 48.

44. $x = t + \frac{1}{t}, y = t - \frac{1}{t}$

Solution:

Given,

$$x = t + \frac{1}{t}, y = t - \frac{1}{t}$$

Differentiating both the parametric functions w.r.t θ

$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$$

$$\text{Thus, } \frac{dy}{dx} = \frac{t^2 + 1}{t^2 - 1}$$

45. $x = e^{\theta} \left(\theta + \frac{1}{\theta} \right), y = e^{-\theta} \left(\theta - \frac{1}{\theta} \right)$

Solution:

Given,

$$x = e^{\theta} \left(\theta + \frac{1}{\theta} \right), y = e^{-\theta} \left(\theta - \frac{1}{\theta} \right)$$

Differentiating both the parametric functions w.r.t. θ .

$$\frac{dx}{d\theta} = e^{\theta} \left(1 - \frac{1}{\theta^2} \right) + \left(\theta + \frac{1}{\theta} \right) \cdot e^{\theta}$$

$$\begin{aligned} \frac{dx}{d\theta} &= e^{\theta} \left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta} \right) \Rightarrow e^{\theta} \left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2} \right) \\ &= \frac{e^{\theta} (\theta^3 + \theta^2 + \theta - 1)}{\theta^2} \end{aligned}$$

$$y = e^{-\theta} \left(\theta - \frac{1}{\theta} \right)$$

$$\frac{dy}{d\theta} = e^{-\theta} \left(1 + \frac{1}{\theta^2} \right) + \left(\theta - \frac{1}{\theta} \right) \cdot (-e^{-\theta})$$

$$\begin{aligned} \frac{dy}{d\theta} &= e^{-\theta} \left(1 + \frac{1}{\theta^2} - \theta + \frac{1}{\theta} \right) \Rightarrow e^{-\theta} \left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2} \right) \\ &= e^{-\theta} \frac{(-\theta^3 + \theta^2 + \theta + 1)}{\theta^2} \end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta} \left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^2} \right)}{e^{\theta} \left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2} \right)} \\ &= e^{-2\theta} \left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1} \right) \\ \text{Thus, } \frac{dy}{dx} &= e^{-2\theta} \left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1} \right).\end{aligned}$$

46. $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$.

Solution:

Given, $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$.

Differentiating both the parametric functions w.r.t. θ

$$\begin{aligned}\frac{dx}{d\theta} &= -3\sin\theta - 6\cos^2\theta \cdot \frac{d}{d\theta}(\cos\theta) \\ &= -3\sin\theta - 6\cos^2\theta \cdot (-\sin\theta) \\ &= -3\sin\theta + 6\cos^2\theta \cdot \sin\theta \\ \frac{dy}{d\theta} &= 3\cos\theta - 6\sin^2\theta \cdot \frac{d}{d\theta}(\sin\theta) \\ &= 3\cos\theta - 6\sin^2\theta \cdot \cos\theta \\ \therefore \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta \cos\theta}{-3\sin\theta + 6\cos^2\theta \cdot \sin\theta} \\ \Rightarrow \frac{dy}{dx} &= \frac{\cos\theta(3 - 6\sin^2\theta)}{\sin\theta(-3 + 6\cos^2\theta)} = \frac{\cos\theta[3 - 6(1 - \cos^2\theta)]}{\sin\theta[-3 + 6\cos^2\theta]} \\ &= \cot\theta \left(\frac{3 - 6 + 6\cos^2\theta}{-3 + 6\cos^2\theta} \right) = \cot\theta \left(\frac{-3 + 6\cos^2\theta}{-3 + 6\cos^2\theta} \right) \\ &= \cot\theta \\ \text{Thus, } \frac{dy}{dx} &= \cot\theta.\end{aligned}$$

47. $\sin x = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$.

Solution:

Given,

$$\sin x = 2t/(1+t^2), \tan y = 2t/(1-t^2)$$

Now, taking $\sin x = \frac{2t}{1+t^2}$

Differentiating both sides w.r.t t , we get

$$\cos x \cdot \frac{dx}{dt} = \frac{(1+t^2) \cdot \frac{d}{dt}(2t) - 2t \cdot \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\cos x \cdot \frac{dx}{dt} = \frac{2(1+t^2) - 2t \cdot 2t}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2+2t^2-4t^2}{(1+t^2)^2} \times \frac{1}{\cos x}$$

$$\frac{dx}{dt} = \frac{2-2t^2}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\sin^2 x}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{\frac{(1+t^2)^2-4t^2}{(1+t^2)^2}}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1+t^2}{\sqrt{1+t^4+2t^2-4t^2}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)}{\sqrt{1+t^4-2t^2}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{\sqrt{(1-t^2)^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{(1-t^2)} \Rightarrow \frac{dx}{dt} = \frac{2}{1+t^2}$$

Now taking, $\tan y = \frac{2}{1-t^2}$

Differentiating both sides w.r.t t , we get

$$\frac{d}{dt}(\tan y) = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\sec^2 y \frac{dy}{dt} = \frac{(1-t^2) \cdot \frac{d}{dt}(2t) - 2t \cdot \frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\sec^2 y \frac{dy}{dt} = \frac{(1-t^2) \cdot 2 - 2t \cdot (-2t)}{(1-t^2)^2}$$

$$\begin{aligned}\sec^2 y \frac{dy}{dt} &= \frac{2 - 2t^2 + 4t^2}{(1 - t^2)^2} \\ \frac{dy}{dt} &= \frac{2 + 2t^2}{(1 - t^2)^2} \times \frac{1}{\sec^2 y} \\ \frac{dy}{dt} &= \frac{2(1 + t^2)}{(1 - t^2)^2} \times \frac{1}{1 + \tan^2 y} \\ \frac{dy}{dt} &= \frac{2(1 + t^2)}{(1 - t^2)^2} \times \frac{1}{1 + \left(\frac{2t}{1 - t^2}\right)^2} \\ \frac{dy}{dt} &= \frac{2(1 + t^2)}{(1 - t^2)^2} \times \frac{1}{\frac{(1 - t^2)^2 + 4t^2}{(1 - t^2)^2}} \\ \frac{dy}{dt} &= \frac{2(1 + t^2)}{(1 - t^2)^2} \times \frac{(1 - t^2)^2}{1 + t^2 + 2t^2 + 4t^2} \\ \frac{dy}{dt} &= \frac{2(1 + t^2)}{(1 - t^2)^2} \times \frac{(1 - t^2)^2}{1 + t^4 + 2t^2} \\ \Rightarrow \frac{dy}{dt} &= \frac{2(1 + t^2)}{(1 - t^2)^2} \times \frac{(1 - t^2)^2}{(1 + t^2)^2} \Rightarrow \frac{dy}{dt} = \frac{2}{1 + t^2} \\ \therefore \frac{dy}{dt} &= \frac{dy/dt}{dx/dt} = \frac{\frac{2}{1 + t^2}}{\frac{2}{1 + t^2}} = 1\end{aligned}$$

Thus, $\frac{dy}{dt} = 1$

48. $x = \frac{1 + \log t}{t^2}, \quad y = \frac{3 + 2 \log t}{t}$

Solution:

On differentiating both the given parametric functions w.r.t. t , we have

$$\begin{aligned}\frac{dx}{dt} &= \frac{t^2 \cdot \frac{d}{dt}(1 + \log t) - (1 + \log t) \cdot \frac{d}{dt}(t^2)}{t^4} \\ &= \frac{t^2 \cdot \left(\frac{1}{t}\right) - (1 + \log t) \cdot 2t}{t^4} = \frac{t - (1 + \log t) \cdot 2t}{t^4} \\ &= \frac{t[1 - 2 - 2 \log t]}{t^4} = \frac{-(1 + 2 \log t)}{t^3}\end{aligned}$$

$$y = \frac{3 + 2 \log t}{t}$$

Next,

$$\frac{dy}{dt} = \frac{t \cdot \frac{d}{dt}(3 + 2 \log t) - (3 + 2 \log t) \cdot \frac{d}{dt}(t)}{t^2}$$

$$= \frac{t(2/t) - (3 + 2 \log t) \cdot 1}{t^2}$$

$$= \frac{2 - 3 - 2 \log t}{t^2} = \frac{-(1 + 2 \log t)}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-(1 + 2 \log t)}{t^2}}{\frac{t^3}{t^2}} = \frac{-(1 + 2 \log t)}{t^3} = t$$

Thus, $\frac{dy}{dx} = t$.

49. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $dy/dx = -y \log x / x \log y$.

Solution:

Given,

$$x = e^{\cos 2t} \text{ and } y = e^{\sin 2t}$$

So, $\cos 2t = \log x$ and $\sin 2t = \log y$

Now, differentiating both the parameter functions w.r.t t , we have

$$\begin{aligned} \frac{dx}{dt} &= e^{\cos 2t} \cdot \frac{d}{dt}(\cos 2t) = e^{\cos 2t} (-\sin 2t) \cdot \frac{d}{dt}(2t) \\ &= -e^{\cos 2t} \cdot \sin 2t \cdot 2 = -2e^{\cos 2t} \cdot \sin 2t \end{aligned}$$

$$\text{Now, } y = e^{\sin 2t}$$

$$\begin{aligned} \frac{dy}{dt} &= e^{\sin 2t} \cdot \frac{d}{dt}(\sin 2t) = e^{\sin 2t} \cdot \cos 2t \cdot \frac{d}{dt}(2t) \\ &= e^{\sin 2t} \cdot \cos 2t \cdot 2 = 2e^{\sin 2t} \cdot \cos 2t \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t} \cdot \cos 2t}{-2e^{\cos 2t} \cdot \sin 2t} = \frac{e^{\sin 2t} \cdot \cos 2t}{-e^{\cos 2t} \cdot \sin 2t} = \frac{y \cos 2t}{-x \sin 2t} \\ &= \frac{y \log x}{-x \log y} \quad \left[\begin{array}{l} \because \cos 2t = \log x \\ \sin 2t = \log y \end{array} \right] \end{aligned}$$

$$\text{Thus, } \frac{dy}{dx} = -\frac{y \log x}{x \log y}.$$

50. If $x = a \sin 2t (1 + \cos 2t)$ and $y = b \cos 2t (1 - \cos 2t)$, show that

Solution:

$$\left(\frac{dy}{dx} \right)_{\text{at } t = \frac{\pi}{4}} = \frac{b}{a}$$

Given,

$$x = a \sin 2t (1 + \cos 2t) \text{ and } y = b \cos 2t (1 - \cos 2t)$$

Differentiating both the parametric equations w.r.t t , we have

$$\begin{aligned}\frac{dx}{dt} &= a \left[\sin 2t \cdot \frac{d}{dt}(1 + \cos 2t) + (1 + \cos 2t) \cdot \frac{d}{dt} \sin 2t \right] \\ &= a [\sin 2t \cdot (-\sin 2t) \cdot 2 + (1 + \cos 2t)(\cos 2t) \cdot 2] \\ &= a [-2 \sin^2 2t + 2 \cos 2t + 2 \cos^2 2t] \\ &= a [2(\cos^2 2t - \sin^2 2t) + 2 \cos 2t] \\ &= a [2 \cos 4t + 2 \cos 2t] \quad [\because \cos 2x = \cos^2 x - \sin^2 x] \\ &= 2a [\cos 4t + \cos 2t]\end{aligned}$$

Now, $y = b \cos 2t (1 - \cos 2t)$

$$\begin{aligned}\frac{dy}{dt} &= b \left[\cos 2t \cdot \frac{d}{dt}(1 - \cos 2t) + (1 - \cos 2t) \cdot \frac{d}{dt}(\cos 2t) \right] \\ &= b [\cos 2t \cdot \sin 2t \cdot 2 + (1 - \cos 2t) \cdot (-\sin 2t) \cdot 2] \\ &= b [2 \sin 2t \cdot \cos 2t - 2 \sin 2t + 2 \sin 2t \cos 2t] \\ &= b [\sin 4t - 2 \sin 2t + \sin 4t] \quad [\because \sin 2x = 2 \sin x \cos x] \\ &= b [2 \sin 4t - 2 \sin 2t] = 2b (\sin 4t - \sin 2t)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b [\sin 4t - \sin 2t]}{2a [\cos 4t + \cos 2t]} = \frac{b}{a} \left[\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t} \right]$$

Putting, $t = \frac{\pi}{4}$ we get

$$\begin{aligned}\left(\frac{dy}{dx} \right)_{at t = \frac{\pi}{4}} &= \frac{b}{a} \left[\frac{\sin 4\left(\frac{\pi}{4}\right) - \sin 2\left(\frac{\pi}{4}\right)}{\cos 4\left(\frac{\pi}{4}\right) + \cos 2\left(\frac{\pi}{4}\right)} \right] = \frac{b}{a} \left[\frac{\sin \pi - \sin \frac{\pi}{2}}{\cos \pi + \cos \frac{\pi}{2}} \right] \\ &= \frac{b}{a} \left[\frac{0 - 1}{-1 + 0} \right] = \frac{b}{a} \left(\frac{-1}{-1} \right) = \frac{b}{a}\end{aligned}$$

Thus, $\left(\frac{dy}{dx} \right)_{at t = \frac{\pi}{4}} = \frac{b}{a}$.

51. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.
Solution:

Given,

$$x = 3\sin t - \sin 3t, y = 3\cos t - \cos 3t$$

Now, differentiating both the parametric functions w.r.t t , we have

$$\begin{aligned}\frac{dx}{dt} &= 3 \cos t - \cos 3t \cdot 3 = 3(\cos t - \cos 3t) \\ \frac{dy}{dt} &= -3 \sin t + \sin 3t \cdot 3 = 3(-\sin t + \sin 3t)\end{aligned}$$

$$\text{So, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(-\sin t + \sin 3t)}{3(\cos t - \cos 3t)} = \frac{-\sin t + \sin 3t}{\cos t - \cos 3t}$$

$$\text{Putting, } t = \frac{\pi}{3}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-\sin \frac{\pi}{3} + \sin 3\left(\frac{\pi}{3}\right)}{\cos \frac{\pi}{3} - \cos 3\left(\frac{\pi}{3}\right)} \\ &= \frac{-\frac{\sqrt{3}}{2} + \sin \pi}{\frac{1}{2} - \cos \pi} = \frac{-\frac{\sqrt{3}}{2} + 0}{\frac{1}{2} - (-1)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2} + 1} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{-1}{\sqrt{3}} \end{aligned}$$

$$\text{Thus, } \frac{dy}{dx} = \frac{-1}{\sqrt{3}}.$$

52. Differentiate $x/\sin x$ w.r.t $\sin x$.

Solution:

$$\text{Let } y = \frac{x}{\sin x} \quad \text{and} \quad z = \sin x.$$

Differentiating both the parametric functions w.r.t. x ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin x \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\sin x)}{(\sin x)^2} \\ &= \frac{\sin x \cdot 1 - x \cdot \cos x}{\sin^2 x} = \frac{\sin x - x \cos x}{\sin^2 x} \end{aligned}$$

$$\frac{dz}{dx} = \cos x$$

$$\begin{aligned} \text{Now, } \frac{dy}{dz} &= \frac{dy/dx}{dz/dx} = \frac{\frac{\sin x - x \cos x}{\sin^2 x}}{\cos x} = \frac{\sin x - x \cos x}{\sin^2 x \cos x} \\ &= \frac{\sin x}{\sin^2 x \cos x} - \frac{x \cos x}{\sin^2 x \cos x} \\ &= \frac{\tan x}{\sin^2 x} - \frac{x}{\sin^2 x} = \frac{\tan x - x}{\sin^2 x} \end{aligned}$$

$$\text{Thus, } \frac{dy}{dz} = \frac{\tan x - x}{\sin^2 x}.$$

53. Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ w.r.t $\tan^{-1} x$ when $x \neq 0$.

Solution:

Let $y = \tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ and $z = \tan^{-1} x$

Now, put $x = \tan \theta$

$\therefore y = \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right)$ and $z = \tan^{-1}(\tan \theta) = \theta$.

So, $\tan \left(\frac{\sqrt{\sec \theta}-1}{\tan \theta} \right) = \tan^{-1} \left(\frac{\sec \theta-1}{\tan \theta} \right)$

$$\tan^{-1} \left(\frac{\frac{1}{\cos \theta}-1}{\frac{\sin \theta}{\cos \theta}} \right) = \tan^{-1} \left(\frac{1-\cos \theta}{\sin \theta} \right)$$

$$\tan^{-1} \left(\frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2} \right) = \tan^{-1} \left(\frac{\sin \theta/2}{\cos \theta/2} \right)$$

$$\Rightarrow y = \tan^{-1} \left(\tan \frac{\theta}{2} \right) \Rightarrow y = \frac{\theta}{2}$$

Differentiating both parametric functions w.r.t. θ

$$\frac{dy}{d\theta} = \frac{1}{2} \cdot \frac{d}{d\theta}(\theta) \quad \text{and} \quad \frac{dz}{d\theta} = \frac{d}{d\theta}(\theta)$$

$$= \frac{1}{2} \cdot 1 = \frac{1}{2} \quad \text{and} \quad \frac{dz}{d\theta} = 1$$

Thus, $\frac{dy}{dz} = \frac{dy/d\theta}{dz/d\theta} = \frac{1/2}{1} = \frac{1}{2}$.