

**Short Answer**

**(S.A.) Verify the following**

1.  $\int \frac{2x-1}{2x+3} dx = x - \log |(2x+3)^2| + C$

**Solution:**

$$\begin{aligned} \text{L.H.S.} &= \int \frac{2x-1}{2x+3} dx \\ &= \int \left( 1 - \frac{4}{2x+3} \right) dx \quad \text{[Dividing the numerator by the denominator]} \\ &= \int 1 \cdot dx - 4 \int \frac{1}{2x+3} dx = \int 1 \cdot dx - \frac{4}{2} \int \frac{1}{x + \frac{3}{2}} dx \\ &= \int 1 \cdot dx - 2 \int \frac{1}{x + \frac{3}{2}} dx = x - 2 \log \left| x + \frac{3}{2} \right| + C \\ &= x - 2 \log \left| \frac{2x+3}{2} \right| + C = x - \log \left| \left( \frac{2x+3}{2} \right)^2 \right| + C \quad [\because n \log m = \log m^n] \\ &= x - \log |(2x+3)^2| - \log 2^2 + C \\ &= x - \log |(2x+3)^2| + C_1 \Rightarrow \text{R.H.S.} \quad [\text{where } C_1 = C - \log 2^2] \\ \text{L.H.S.} &= \text{R.H.S.} \end{aligned}$$

- Hence Proved

2.  $\int \frac{2x+3}{x^2+3x} dx = \log |x^2+3x| + C$

**Solution:**

$$\begin{aligned} \text{L.H.S.} &= \int \frac{2x+3}{x^2+3x} dx \\ \text{Putting, } x^2+3x &= t \\ \text{So, } (2x+3) dx &= dt \\ \Rightarrow \int \frac{dt}{t} &= \log |t| \Rightarrow \log |x^2+3x| + C = \text{R.H.S.} \\ \text{L.H.S.} &= \text{R.H.S.} \\ &\text{- Hence proved.} \end{aligned}$$

**Evaluate the following:**

3.  $\int \frac{(x^2 + 2) dx}{x + 1}$

**Solution:**

Let  $I = \int \frac{x^2 + 2}{x + 1} dx$

So,  $I = \int \left[ (x - 1) + \frac{3}{x + 1} \right] dx$   
 $= \int (x - 1) dx + 3 \int \frac{1}{x + 1} dx$   
 $= \frac{x^2}{2} - x + 3 \log |x + 1| + C$

Performing long division of the given integral, we have

$$\begin{array}{r} x + 1 \overline{) x^2 + 2} \\ \underline{(-) x^2 + x} \phantom{0} \\ -x + 2 \\ \underline{(+ ) x + 1} \\ 3 \end{array}$$

Therefore,

$$I = \frac{x^2}{2} - x + 3 \log |x + 1| + C$$

4.  $\int \frac{e^{6 \log x} - e^{5 \log x}}{e^{4 \log x} - e^{3 \log x}} dx$

**Solution:**

Let  $I = \int \frac{e^{6 \log x} - e^{5 \log x}}{e^{4 \log x} - e^{3 \log x}} dx = \int \frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}} dx$   
 $= \int \frac{x^6 - x^5}{x^4 - x^3} dx$  (Using properties of logarithms)  
 $= \int \frac{x^2(x^4 - x^3)}{x^4 - x^3} dx = \int x^2 dx = \frac{1}{3} x^3 + C$

Therefore,

$$I = \int x^2 dx = \frac{1}{3} x^3 + C$$

5.  $\int \frac{(1 + \cos x)}{x + \sin x} dx$

**Solution:**

$$I = \int \frac{1 + \cos x}{x + \sin x} dx$$

Putting,  $x + \sin x = t \Rightarrow (1 + \cos x) dx = dt$

So,  $I = \int \frac{dt}{t} = \log |t| = \log |x + \sin x| + C$

Therefore,

$I = \log |t| = \log |x + \sin x| + C$

6.  $\int \frac{dx}{1 + \cos x}$

**Solution:**

Let  $I = \int \frac{dx}{1 + \cos x} = \int \frac{dx}{2 \cos^2 \frac{x}{2}} \left[ \because 1 + \cos x = 2 \cos^2 \frac{x}{2} \right]$   
 $= \frac{1}{2} \int \sec^2 \frac{x}{2} dx = \frac{1}{2} \cdot 2 \tan \frac{x}{2} + C = \tan \frac{x}{2} + C$

Therefore,

$I = \tan \frac{x}{2} + C$

7.  $\int \tan^2 x \sec^4 x dx$

**Solution:**

Let  $I = \int \tan^2 x \cdot \sec^4 x dx$   
 $= \int \tan^2 x \sec^2 x \cdot \sec^2 x dx = \int \tan^2 x (1 + \tan^2 x) \cdot \sec^2 x dx$

Putting,  $\tan x = t, \Rightarrow \sec^2 x dx = dt$

$I = \int t^2 (1 + t^2) dt = \int (t^2 + t^4) dt = \int t^2 dt + \int t^4 dt$   
 $= \frac{1}{3} t^3 + \frac{1}{5} t^5 = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$

Therefore,

$I = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$

8.  $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

**Solution:**

$$\begin{aligned}
 \text{Let } I &= \int \frac{\sin x + \cos x}{\sqrt{1 + 2 \sin x \cos x}} dx \\
 &= \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}} dx \\
 &= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx = \int \frac{\sin x + \cos x}{\sin x + \cos x} dx \\
 &= \int 1 dx \\
 &= x + C, \text{ where } C \text{ is a constant}
 \end{aligned}$$

Therefore,

$$I = x + C$$

9.  $\int \sqrt{1 + \sin x} dx$

**Solution:**

$$\begin{aligned}
 \text{Let } I &= \int \sqrt{1 + \sin x} dx \\
 &= \int \sqrt{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}\right)} dx \\
 &= \int \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} dx = \int \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) dx \\
 &= \int \sin \frac{x}{2} dx + \int \cos \frac{x}{2} dx = -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} + C \\
 &= 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + C, \text{ where } C \text{ is a constant}
 \end{aligned}$$

Therefore,

$$I = 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + C$$

10.  $\int \frac{x}{\sqrt{x+1}} dx$  (Hint : Put  $\sqrt{x} = z$ )

**Solution:**

$$\text{Let, } I = \int \frac{x}{\sqrt{x+1}} dx$$

$$\text{Put } \sqrt{x} = t \Rightarrow x = t^2 \therefore dx = 2t \cdot dt$$

$$\begin{aligned}
 \text{So, } I &= \int \frac{t^2 \cdot 2t \cdot dt}{t+1} = 2 \int \frac{t^3}{t+1} dt = 2 \int \frac{t^3 + 1 - 1}{t+1} dt \\
 &= 2 \int \frac{t^3 + 1}{t+1} dt - 2 \int \frac{1}{t+1} dt
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \int \frac{(t+1)(t^2-t+1)}{t+1} dt - 2 \int \frac{1}{t+1} dt \\
 &= 2 \int (t^2-t+1) dt - 2 \int \frac{1}{t+1} dt \\
 &= 2 \left[ \frac{t^3}{3} - \frac{t^2}{2} + t \right] - 2 \log |t+1| \\
 &= 2 \left[ \frac{x^{3/2}}{3} - \frac{x}{2} + \sqrt{x} \right] - 2 \log |\sqrt{x}+1| + C \\
 &= 2 \left[ \frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log |\sqrt{x}+1| \right] + C
 \end{aligned}$$

Therefore,

$$I = 2 \left[ \frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log |\sqrt{x}+1| \right] + C$$

11.  $\int \sqrt{\frac{a+x}{a-x}} dx$

**Solution:**

$$\begin{aligned}
 \text{Let, } I &= \int \sqrt{\frac{a+x}{a-x}} dx \\
 &= \int \sqrt{\frac{a+x}{a-x}} \times \frac{a+x}{a+x} dx = \int \frac{a+x}{\sqrt{(a-x)(a+x)}} dx \\
 &= \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \frac{a}{\sqrt{a^2-x^2}} dx + \int \frac{x}{\sqrt{a^2-x^2}} dx
 \end{aligned}$$

Let's consider,  $I = I_1 + I_2$

$$\text{Now, } I_1 = \int \frac{a}{\sqrt{a^2-x^2}} dx = a \cdot \sin^{-1} \frac{x}{a} + C_1$$

$$\text{And, } I_2 = \int \frac{x}{\sqrt{a^2-x^2}} dx$$

$$\text{Putting, } a^2 - x^2 = t \Rightarrow -2x dx = dt$$

$$x dx = \frac{dt}{-2}$$

$$\text{Hence, } I_2 = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \times 2\sqrt{t} = -\sqrt{a^2-x^2} + C_2$$

$$\text{As, } I = I_1 + I_2$$

$$= a \sin^{-1} \frac{x}{a} + C_1 - \sqrt{a^2-x^2} + C_2$$

$$\therefore I = a \sin^{-1} \frac{x}{a} - \sqrt{a^2 - x^2} + (C_1 + C_2)$$

Therefore,  $I = a \sin^{-1} \frac{x}{a} - \sqrt{a^2 - x^2} + C$  [ $C = C_1 + C_2$ ]

12.  $\int \frac{x^{\frac{1}{2}}}{1+x^4} dx$  (Hint : Put  $x = z^4$ )

**Solution:**

Let,  $I = \int \frac{x^{1/2}}{1+x^{3/4}} dx$

Putting,  $x = t^4 \Rightarrow dx = 4t^3 dt$

$$= \int \frac{t^2 \cdot 4t^3}{1+t^3} dt = 4 \int \frac{t^5}{1+t^3} dt$$

$$= 4 \int \left( t^2 - \frac{t^2}{t^3+1} \right) dt = 4 \int t^2 dt - 4 \int \frac{t^2}{t^3+1} dt$$

Let's consider,  $I = I_1 - I_2$

Now,  $I_1 = 4 \int t^2 dt = 4 \cdot \frac{t^3}{3} + C_1 = \frac{4}{3} x^{3/4} + C_1$

$$I_2 = 4 \int \frac{t^2}{t^3+1} dt$$

Putting,  $t^3 + 1 = z \Rightarrow 3t^2 dt = dz$

$$t^2 dt = \frac{1}{3} dz$$

So,  $I_2 = \frac{4}{3} \int \frac{dz}{z} = \frac{4}{3} \log |z| + C_2 = \frac{4}{3} \log |t^3 + 1| + C_2$

$$= \frac{4}{3} \log |(x)^{3/4} + 1| + C_2$$

As,  $I = I_1 - I_2$

$$= \frac{4}{3} x^{3/4} + C_1 - \frac{4}{3} \log |(x)^{3/4} + 1| - C_2$$

$$= \frac{4}{3} [x^{3/4} - \log |(x)^{3/4} + 1|] + C_1 - C_2$$

Therefore,  $I = \frac{4}{3} [x^{3/4} - \log |(x)^{3/4} + 1|] + C$  [ $\because C = C_1 - C_2$ ]

13.  $\int \frac{\sqrt{1+x^2}}{x^4} dx$

**Solution:**

$$\begin{aligned}\text{Let, } I &= \int \frac{\sqrt{1+x^2}}{x^4} dx \\ &= \int \frac{\sqrt{1+x^2}}{x^2} \cdot \frac{1}{x^3} dx = \int \sqrt{\frac{1}{x^2} + 1} \cdot \frac{1}{x^3} dx\end{aligned}$$

Taking,  $\frac{1}{x^2} + 1 = t^2$

$$\begin{aligned}\text{So, } \frac{-2}{x^3} dx &= 2t dt \Rightarrow \frac{dx}{x^3} = -t dt \\ \therefore I &= \int t(-t dt) = -\int t^2 dt = -\frac{1}{3}t^3 + C\end{aligned}$$

Therefore,  $I = -\frac{1}{3}\left(\frac{1}{x^2} + 1\right)^{3/2} + C$

$$\int \frac{dx}{\sqrt{16-9x^2}}$$

14.

**Solution:**

$$\begin{aligned}\text{Let, } I &= \int \frac{dx}{\sqrt{16-9x^2}} \\ &= \frac{1}{3} \int \frac{dx}{\sqrt{\frac{16}{9}-x^2}} = \frac{1}{3} \int \frac{dx}{\sqrt{\left(\frac{4}{3}\right)^2-x^2}} \\ &= \frac{1}{3} \sin^{-1} \frac{x}{4/3} + C \quad \left[ \because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C \right] \\ &= \frac{1}{3} \sin^{-1} \frac{3x}{4} + C\end{aligned}$$

Therefore,  $I = \frac{1}{3} \sin^{-1} \frac{3x}{4} + C.$

$$\int \frac{dt}{\sqrt{3t-2t^2}}$$

15.

**Solution:**

$$\text{Let, } I = \int \frac{dt}{\sqrt{3t-2t^2}} = \int \frac{dt}{\sqrt{-2\left(t^2-\frac{3}{2}t\right)}}$$



$$\begin{aligned}
 &= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left(t^2 - \frac{3}{2}t + \frac{9}{16} - \frac{9}{16}\right)}} \quad [\text{Making perfect square}] \\
 &= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left[\left(t - \frac{3}{4}\right)^2 - \frac{9}{16}\right]}} = \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\frac{9}{16} - \left(t - \frac{3}{4}\right)^2}} \\
 &= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}} \cdot \sin^{-1} \frac{t - \frac{3}{4}}{\frac{3}{4}} + C \\
 &= \frac{1}{\sqrt{2}} \sin^{-1} \frac{4t - 3}{3} + C
 \end{aligned}$$

Therefore,  $I = \frac{1}{\sqrt{2}} \sin^{-1} \left( \frac{4t - 3}{3} \right) + C.$

16.  $\int \frac{3x-1}{\sqrt{x^2+9}} dx$

**Solution:**

Let,  $I = \int \frac{3x-1}{\sqrt{x^2+9}} dx = \int \frac{3x}{\sqrt{x^2+9}} dx - \int \frac{1}{\sqrt{x^2+9}} dx$

$I = I_1 - I_2$

Now,  $I_1 = \int \frac{3x}{\sqrt{x^2+9}} dx$

Putting,  $x^2 + 9 = t \Rightarrow 2x dx = dt$

So,  $x dx = \frac{1}{2} dt$

$I_1 = \frac{3}{2} \int \frac{dt}{\sqrt{t}} = \frac{3}{2} \cdot 2\sqrt{t} + C_1 = 3\sqrt{x^2+9} + C_1$

And,

$I_2 = \int \frac{1}{\sqrt{x^2+9}} dx = \int \frac{1}{\sqrt{x^2+(3)^2}} dx = \log |x + \sqrt{x^2+(3)^2}| + C_2$

$\left[ \because \int \frac{1}{\sqrt{x^2+a^2}} dx = \log |x + \sqrt{x^2+a^2}| + C \right]$

$= \log |x + \sqrt{x^2+9}| + C_2$

$\therefore I = I_1 - I_2$

$= 3\sqrt{x^2+9} + C_1 - \log |x + \sqrt{x^2+9}| - C_2$



$$= 3\sqrt{x^2 + 9} - \log |x + \sqrt{x^2 + 9}| + (C_1 - C_2)$$

Therefore,  $I = 3\sqrt{x^2 + 9} - \log |x + \sqrt{x^2 + 9}| + C$

17.  $\int \sqrt{5 - 2x + x^2} dx$

**Solution:**

$$\begin{aligned} \text{Let } I &= \int \sqrt{5 - 2x + x^2} dx = \int \sqrt{x^2 - 2x + 5} dx \\ &= \int \sqrt{x^2 - 2x + 1 - 1 + 5} dx \quad (\text{Making perfect square}) \\ &= \int \sqrt{(x-1)^2 + 4} dx = \int \sqrt{(x-1)^2 + (2)^2} dx \\ &= \frac{x-1}{2} \sqrt{(x-1)^2 + (2)^2} + \frac{4}{2} \log |(x-1) + \sqrt{(x-1)^2 + (2)^2}| + C \\ &\quad \left[ \because \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \left\{ \log |x + \sqrt{x^2 + a^2}| \right\} + C \right] \\ &= \frac{x-1}{2} \sqrt{x^2 + 1 - 2x + 4} + 2 \log |(x-1) + \sqrt{x^2 + 1 - 2x + 4}| + C \\ &= \frac{x-1}{2} \sqrt{x^2 - 2x + 5} + 2 \log |(x-1) + \sqrt{x^2 - 2x + 5}| + C \end{aligned}$$

Therefore,

$$I = \frac{x-1}{2} \sqrt{x^2 - 2x + 5} + 2 \log |(x-1) + \sqrt{x^2 - 2x + 5}| + C$$

18.  $\int \frac{x}{x^4 - 1} dx$

**Solution:**

Let,  $I = \int \frac{x}{x^4 - 1} dx$

Putting,  $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$

$$\begin{aligned} \frac{1}{2} \int \frac{dt}{t^2 - 1} &= \frac{1}{2} \int \frac{dt}{t^2 - (1)^2} = \frac{1}{2} \cdot \frac{1}{2 \cdot 1} \log \left| \frac{t-1}{t+1} \right| + C \\ &\quad \left[ \because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right] \\ &= \frac{1}{4} \log \left| \frac{x^2 - 1}{x^2 + 1} \right| + C \end{aligned}$$

Therefore,  $I = \frac{1}{4} \log \left| \frac{x^2 - 1}{x^2 + 1} \right| + C$

19.  $\int \frac{x^2}{1-x^4} dx$  put  $x^2 = t$

**Solution:**

Let  $I = \int \frac{x^2}{1-x^4} dx = \int \frac{x^2}{(1-x^2)(1+x^2)} dx$

Putting,  $x^2 = t$  for the purpose of partial fractions.

We get  $\frac{t}{(1-t)(1+t)}$

Resolving into partial fraction, we have

$$\frac{t}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$$

[where  $A$  and  $B$  are arbitrary constants]

So,  $\frac{t}{(1-t)(1+t)} = \frac{A(1+t) + B(1-t)}{(1-t)(1+t)}$

$\Rightarrow t = A + At + B - Bt$

Comparing the like terms, we get  $A - B = 1$  and  $A + B = 0$

On solving the above equations, we have

$A = \frac{1}{2}$  and  $B = -\frac{1}{2}$

Now,  $I = \int \frac{1/2}{1-x^2} dx + \int \frac{-1/2}{1+x^2} dx$  (Putting  $t = x^2$ )

$$= \frac{1}{2} \cdot \frac{1}{2.1} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C$$

$$= \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C$$

Therefore,  $I = \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C.$

20.  $\int \sqrt{2ax - x^2} dx$

**20.**

**Solution:**

$$\begin{aligned}
 \text{Let } I &= \int \sqrt{2ax - x^2} \, dx \\
 &= \int \sqrt{-(x^2 - 2ax)} \, dx = \int \sqrt{-(x^2 - 2ax + a^2 - a^2)} \, dx \\
 &= \int \sqrt{-(x-a)^2 + a^2} \, dx = \int \sqrt{a^2 - (x-a)^2} \, dx \\
 &= \frac{x-a}{2} \sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x-a}{a} \right) + C \\
 &\quad \left[ \because \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \right] \\
 &= \frac{x-a}{2} \sqrt{a^2 - (x^2 - 2ax + a^2)} + \frac{a^2}{2} \sin^{-1} \left( \frac{x-a}{a} \right) + C \\
 &= \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x-a}{a} \right) + C \\
 \text{Therefore, } I &= \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x-a}{a} \right) + C.
 \end{aligned}$$

21. 
$$\int \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$$

**Solution:**

$$\text{Let } I = \int \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$$

$$\text{Putting, } x = \sin \theta \Rightarrow dx = \cos \theta \, d\theta$$

$$\begin{aligned}
 I &= \int \frac{\sin^{-1}(\sin \theta)}{(1 - \sin^2 \theta)^{\frac{3}{2}}} \cdot \cos \theta \, d\theta \\
 &= \int \frac{\theta \cdot \cos \theta \, d\theta}{(\cos^2 \theta)^{\frac{3}{2}}} = \int \frac{\theta \cdot \cos \theta}{\cos^3 \theta} d\theta \\
 &= \int \frac{\theta}{\cos^2 \theta} d\theta = \int \theta \sec^2 \theta \, d\theta \\
 &= \theta \cdot \int \sec^2 \theta \, d\theta - \int \left( D(\theta) \cdot \int \sec^2 \theta \, d\theta \right) d\theta \\
 &\quad \left[ \because \int u \cdot v \, dx = u \cdot \int v \, dx - \int \left( D(u) \int v \, dx \right) dx + C \right] \\
 &= \theta \cdot \tan \theta - \int 1 \cdot \tan \theta \, d\theta \\
 &= \theta \cdot \tan \theta - \log \sec \theta + C
 \end{aligned}$$

$$= \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} - \log |\sqrt{1-x^2}| + C$$

$$\left[ \begin{array}{l} \text{when } x = \sin \theta \\ \therefore \tan \theta = \frac{x}{\sqrt{1-x^2}} \text{ and } \sec \theta = \sqrt{1-x^2} \end{array} \right]$$

Therefore,  $I = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - \log |\sqrt{1-x^2}| + C$

22.  $\int \frac{(\cos 5x + \cos 4x)}{1 - 2 \cos 3x} dx$

**Solution:**

$$\begin{aligned} \text{Let } I &= \int \frac{\cos 5x + \cos 4x}{1 - 2 \cos 3x} dx = \int \frac{2 \cos \frac{5x+4x}{2} \cdot \cos \frac{5x-4x}{2}}{1 - 2 \left( 2 \cos^2 \frac{3x}{2} - 1 \right)} dx \\ &= \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{1 - 4 \cos^2 \frac{3x}{2} + 2} dx = \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{3 - 4 \cos^2 \frac{3x}{2}} dx \quad [\text{Using trigonometric identities}] \\ &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{4 \cos^2 \frac{3x}{2} - 3} dx = - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{4 \cos^3 \frac{3x}{2} - 3 \cos \frac{3x}{2}} dx \\ &\quad \left[ \text{Multiplying and dividing by } \cos \frac{3x}{2} \right] \\ &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos 3 \cdot \frac{3x}{2}} dx \\ &\quad [\because \cos 3x = 4 \cos^3 x - 3 \cos x] \\ &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos \frac{9x}{2}} dx = - \int 2 \cos \frac{3x}{2} \cdot \cos \frac{x}{2} dx \\ &= - \int \left[ \cos \left( \frac{3x}{2} + \frac{x}{2} \right) + \cos \left( \frac{3x}{2} - \frac{x}{2} \right) \right] dx \\ &= - \int (\cos 2x + \cos x) dx \\ &\quad [\because 2 \cos A \cos B = \cos (A+B) + \cos (A-B)] \end{aligned}$$

$$= -\int \cos 2x \, dx - \int \cos x \, dx = -\frac{1}{2} \sin 2x - \sin x + C$$

Therefore,  $I = -\left[\frac{1}{2} \sin 2x + \sin x\right] + C.$

$$\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

**23.**

**Solution:**

$$\begin{aligned} \text{Let } I &= \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx \\ &\quad [\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)] \\ &= \int \frac{(1)^3 - 3\sin^2 x \cos^2 x (1)}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \left( \frac{1}{\sin^2 x \cos^2 x} - \frac{3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} \right) dx \\ &= \int \left( \frac{1}{\sin^2 x \cos^2 x} - 3 \right) dx = \int \left( \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3 \right) dx \\ &= \int \left[ \left( \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) - 3 \right] dx \\ &= \int (\sec^2 x + \operatorname{cosec}^2 x - 3) dx \\ &= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx - 3 \int 1 \, dx \\ &= \tan x - \cot x - 3x + C \end{aligned}$$

Therefore,  $I = \tan x - \cot x - 3x + C.$

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$$

**24.**

**Solution:**

$$\text{Let } I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{x^{1/2}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} dx$$

Putting,  $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2} dx = dt \Rightarrow x^{1/2} dx = \frac{2}{3} dt$

Now, 
$$1 = \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - (t)^2}}$$

$$= \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + C = \frac{2}{3} \sin^{-1} \left( \frac{x^{3/2}}{a^{3/2}} \right) + C$$

Therefore,  $1 = \frac{2}{3} \sin^{-1} \left( \frac{x}{a} \right)^{3/2} + C.$

25. 
$$\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$$

**Solution:**

Let  $I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

$$= \int \frac{2 \sin \frac{x+2x}{2} \cdot \sin \left( \frac{2x-x}{2} \right)}{2 \sin^2 x/2} dx$$

$$\left[ \because \cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2} \right]$$

$$= \int \frac{2 \sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{2 \sin^2 \frac{x}{2}} dx = \int \frac{\sin \frac{3x}{2}}{\sin \frac{x}{2}} dx = \int \frac{\sin 3\left(\frac{x}{2}\right)}{\sin \frac{x}{2}} dx$$

$$= \int \frac{3 \sin \frac{x}{2} - 4 \sin^3 \frac{x}{2}}{\sin \frac{x}{2}} dx \quad \left[ \text{Since, } \sin 3x = 3 \sin x - 4 \sin^3 x \right]$$

$$= \int \frac{\sin \frac{x}{2} \left( 3 - 4 \sin^2 \frac{x}{2} \right)}{\sin \frac{x}{2}} dx = \int \left( 3 - 4 \sin^2 \frac{x}{2} \right) dx$$

$$= \int [3 - 2(1 - \cos x)] dx \quad \left[ \because 2 \sin^2 \frac{x}{2} = 1 - \cos x \right]$$

$$= \int (3 - 2 + 2 \cos x) dx = \int (1 + 2 \cos x) dx$$

$$= x + 2 \sin x + C$$

Therefore,  $I = x + 2 \sin x + C.$

26. 
$$\int \frac{dx}{x\sqrt{x^4 - 1}} \quad (\text{Hint : Put } x^2 = \sec \theta)$$



**Solution:**

$$\text{Let } I = \int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{x dx}{x^2\sqrt{x^4-1}}$$

Putting,  $x^2 = \sec \theta$

$$\text{So, } 2x dx = \sec \theta \tan \theta d\theta$$

$$x dx = \frac{1}{2} \sec \theta \tan \theta d\theta$$

$$\begin{aligned} \text{Now, } I &= \frac{1}{2} \int \frac{\sec \theta \tan \theta}{\sec \theta \sqrt{\sec^2 \theta - 1}} d\theta \\ &= \frac{1}{2} \int \frac{\sec \theta \tan \theta}{\sec \theta \cdot \tan \theta} d\theta = \frac{1}{2} \int 1 d\theta = \frac{1}{2} \theta + C \\ I &= \frac{1}{2} \sec^{-1} x^2 + C \end{aligned}$$

$$\text{Therefore, } I = \frac{1}{2} \sec^{-1} x^2 + C.$$

**Evaluate the following as limit of sums:**

$$27. \int_0^2 (x^2 + 3) dx$$

**Solution:**

$$\text{Let } I = \int_0^2 (x^2 + 3) dx$$

Using the formula,

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

$$\text{where } h = \frac{b-a}{n}$$

Here, we have  $a = 0$  and  $b = 2$

So,

$$h = \frac{2-0}{n} \quad \therefore nh = 2$$

Here,

$$f(x) = x^2 + 3$$

$$f(0) = 0 + 3 = 3$$

$$f(0+h) = (0+h)^2 + 3 = h^2 + 3$$

$$f(0+2h) = (0+2h)^2 + 3 = 4h^2 + 3$$

.....

.....

$$f(0+(n-1)h) = (0+(n-1)h)^2 + 3(n-1)^2 h^2 + 3$$

Now

$$\begin{aligned}
 \int_0^2 (x^2 + 3) dx &= \lim_{h \rightarrow 0} h \left[ 3 + h^2 + 3 + 4h^2 + 3 + \dots + (n-1)^2 h^2 + 3 \right] \\
 &= \lim_{h \rightarrow 0} h \left[ (3 + 3 + 3 + \dots + n) + \{h^2 + 4h^2 + \dots + (n-1)^2 h^2\} \right] \\
 &= \lim_{h \rightarrow 0} h \left[ 3n + h^2 \{1 + 4 + \dots + (n-1)^2\} \right] \\
 &= \lim_{h \rightarrow 0} h \left[ 3n + h^2 \frac{n(n-1)(2n-1)}{6} \right] \\
 &\quad \left[ \because 1 + 4 + 9 + \dots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[ 3nh + \frac{h^3 n(n-1)(2n-1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[ 3nh + \frac{nh(nh-h)(2nh-h)}{6} \right] \\
 &= \left[ 3 \times 2 + \frac{2(2-0)(2 \times 2 - 0)}{6} \right] \quad \left[ \because nh = 2 \atop h = 0 \right] \\
 &= \left[ 6 + \frac{2 \times 2 \times 4}{6} \right] = 6 + \frac{8}{3} = \frac{26}{3}
 \end{aligned}$$

Therefore,  $\int_0^2 (x^2 + 3) dx = \frac{26}{3}$ .

28.  $\int_0^2 e^x dx$

**Solution:**

Let,  $I = \int_0^2 e^x dx$

Here,  $a = 0$  and  $b = 2 \therefore h = \frac{b-a}{n} \Rightarrow h = \frac{2-0}{n} \therefore nh = 2$

Now,  $f(x) = e^x$

$$f(0) = e^0 = 1$$

$$f(0+h) = e^{0+h} = e^h$$

$$f(0+2h) = e^{0+2h} = e^{2h}$$

$$f(0 + \overline{n-1}h) = e^{0 + (n-1)h} = e^{(n-1)h}$$

Using  $\int_a^b f(x) dx$

$$= \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a + \overline{n-1}h)]$$

$$\therefore \int_0^2 e^x dx = \lim_{h \rightarrow 0} h [1 + e^h + e^{2h} + \dots + e^{(n-1)h}]$$

$$= \lim_{h \rightarrow 0} h \left[ \frac{1(e^{nh} - 1)}{e^h - 1} \right]$$

$$\left[ \because a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \right]$$

$$= \lim_{h \rightarrow 0} \frac{e^{nh} - 1}{e^h - 1} = \frac{e^2 - 1}{1} = e^2 - 1 \left[ \because \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right]$$

Therefore,  $I = e^2 - 1$ .

**Evaluate the following:**

29.  $\int_0^1 \frac{dx}{e^x + e^{-x}}$

**Solution:**

Let  $I = \int_0^1 \frac{dx}{e^x + e^{-x}}$

$$= \int_0^1 \frac{dx}{e^x + \frac{1}{e^x}} = \int_0^1 \frac{dx}{\frac{e^{2x} + 1}{e^x}} = \int_0^1 \frac{e^x dx}{e^{2x} + 1}$$

Putting,  $e^x = t \Rightarrow e^x dx = dt$

Changing the limit, we have

When  $x = 0 \quad \therefore t = e^0 = 1$

When  $x = 1 \quad \therefore t = e^1 = e$

Now,

$$I = \int_1^e \frac{dt}{t^2 + 1} = [\tan^{-1} t]_1^e$$

$$= [\tan^{-1} e - \tan^{-1}(1)] = \tan^{-1} e - \frac{\pi}{4}$$

Therefore,  $I = \tan^{-1} e - \frac{\pi}{4}$

$$\int_0^{\frac{\pi}{2}} \frac{\tan x \, dx}{1 + m^2 \tan^2 x}$$

30.

**Solution:**

$$\begin{aligned} \text{Let, } I &= \int_0^{\pi/2} \frac{\tan x}{1 + m^2 \tan^2 x} dx \\ &= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1 + m^2 \frac{\sin^2 x}{\cos^2 x}} dx = \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + m^2 \sin^2 x}{\cos^2 x}} dx \\ &= \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx = \int_0^{\pi/2} \frac{\sin x \cos x}{1 - \sin^2 x + m^2 \sin^2 x} dx \\ &= \int_0^{\pi/2} \frac{\sin x \cos x}{1 - \sin^2 x (1 - m^2)} dx \end{aligned}$$

Putting,  $\sin^2 x = t$

$$2 \sin x \cos x \, dx = dt$$

$$\sin x \cos x \, dx = \frac{dt}{2}$$

Changing the limits we get,

$$\text{When } x = 0 \quad \therefore t = \sin^2 0 = 0; \quad \text{When } x = \frac{\pi}{2} \quad \therefore t = \sin^2 \frac{\pi}{2} = 1$$

$$\text{Now, } I = \frac{1}{2} \int_0^1 \frac{dt}{1 - (1 - m^2)t}$$

$$\begin{aligned} I &= \frac{1}{2} \int_0^1 \frac{dt}{1 + (m^2 - 1)t} = \frac{1}{2} \left[ \frac{\log [1 + (m^2 - 1)t]}{m^2 - 1} \right]_0^1 \\ &= \frac{1}{2(m^2 - 1)} [\log (1 + m^2 - 1) - \log (1)] = \frac{\log |m^2|}{2(m^2 - 1)} \end{aligned}$$

$$\text{Therefore, } I = \frac{\log |m^2|}{2(m^2 - 1)} = \frac{\log |m|}{m^2 - 1}$$

$$\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$$

31.

**Solution:**

$$\begin{aligned}
 \text{Let } I &= \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} \\
 &= \int_1^2 \frac{dx}{\sqrt{2x - x^2 - 2 + x}} = \int_1^2 \frac{dx}{\sqrt{-x^2 + 3x - 2}} \\
 &= \int_1^2 \frac{dx}{\sqrt{-(x^2 - 3x + 2)}} \\
 &= \int_1^2 \frac{dx}{\sqrt{-(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2)}} \quad [\text{Making perfect square}] \\
 &= \int_1^2 \frac{dx}{\sqrt{-\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}} = \int_1^2 \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{3}{2}\right)^2}} \\
 &= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[ \sin^{-1} \left( \frac{x - \frac{3}{2}}{\frac{1}{2}} \right) \right]_1^2 \\
 &= \left[ \sin^{-1} \left( \frac{2x - 3}{1} \right) \right]_1^2 = \sin^{-1}(4 - 3) - \sin^{-1}(2 - 3) \\
 &= \sin^{-1}(1) - \sin^{-1}(-1) = \sin^{-1}(1) + \sin^{-1}(1) \\
 &= 2 \sin^{-1}(1) = 2 \times \frac{\pi}{2} = \pi
 \end{aligned}$$

Therefore,  $I = \pi$